Equivalence of Algebraic Expressions

Herein we explore an apparently simple concept, taught at the secondary school level: the equivalence of two algebraic expressions. We begin by considering a first example: the equivalence between $(x+1)^2$ and x^2+2x+1. We generally *prove the equivalence of two such expressions in a syntactic manner*, by using symbol manipulation:

$$
\begin{aligned}
(x+1)^2 &= (x+1)(x+1) && \text{by definition of the exponent 2}\\
&= (x+1)x+(x+1)1 && \text{by distributivity}\\
&= (x+1)x+x+1 && \text{because 1 is an identity for multiplication}\\
&= xx+x+x+1 && \text{by distributivity again}\\
&= x^2+x+x+1 && \text{by definition of the exponent 2}\\
&= x^2+2x+1 && \text{regrouping terms in } x
\end{aligned}
$$

However, in secondary school, we do not usually bother with an exhaustive set of rules that can be used in symbol manipulation[1], and we do not always make explicit all the rules that are used. For example, in the preceding manipulations, the associativity of addition (and the related written simplification) is left implied, as is the recourse to arithmetical knowledge[2].

Every mathematics teacher knows that such symbol manipulation is almost meaningless to some students, who will even invent new rules like

$$(x+1)^2 = x^2+1^2 = x^2+1.$$

Such students lose track of the fact that these symbols refer to mathematical objects – numbers in the context of elementary algebra. They forget that the latter chain of equalities has the following meaning: For every number x, all the equalities are verified. In particular, in the case where $x = 1$, they would obtain

$$(1+1)^2 = 1^2+1^2 = 1+1,$$

[1] And we bother even less about the question of completeness, i.e., are our rules sufficient to establish all the equivalences?

[2] We could make explicit the arithmetic rules used as follows:

$$x+x=(1\bullet x)+(1\bullet x)=(1+1)\bullet x=2\bullet x=2x.$$

For the secondary school teacher, this is too much detail and even counterproductive. For the mathematician, it is simply a level of detail among others, which could even be expanded, e.g., by using the Peano axioms.

that is, 4 = 2 = 2, which is evidently false[3].

In general, *we prove the non-equivalence of two expressions in a semantic way*[4], by finding a counterexample. This presupposes reference to a set $\mathcal{E}$ of numbers (natural, integer, relative, decimal, rational, real, or complex) on which operations (addition, multiplication, and perhaps subtraction, division) are defined.

We shall restrict ourselves to single-variable expressions. We just implicitly saw that there are two definitions for the equivalence of two expressions $f(x)$ and $g(x)$, equivalence for which we will use the usual notation $f(x) \equiv g(x)$:

- A *syntactic definition*: $f(x)$ and $g(x)$ are equivalent if and only if we can establish their equality by symbol manipulation, using rules recognized as true for the set $\mathcal{E}$.
- A *semantic definition*: $f(x)$ and $g(x)$ are equivalent if and only if for every element a in $\mathcal{E}$ we have an equality between $f(a)$ and $g(a)$ (we shall refer to this particular definition as Semantic Definition of Equivalence, Version 1).

We have already seen the difficulty involved in making the syntactic definition more precise, as this would require an exhaustive enumeration of all recognized rules. We shall not pursue this direction further. Instead, we choose to consider the semantic definition, which seems less problematic. This definition poses no problem in the case where the expressions $f(x)$ and $g(x)$ are polynomials; but we will see that the situation gets more complex if we accept, within our expressions, operations such as division, roots, or other functions like trigonometric functions.

Let's consider, for example, the following "equivalences":

$$\frac{x-1}{x-1} \equiv 1,\ \sqrt{4x} \equiv 2\sqrt{x} \text{ and } \cos(x) \times \tan(x) \equiv \sin(x).$$

If we apply Version 1 of our Semantic Definition of Equivalence, these "equivalences" are all false, because we can find a counterexample in each case:

$$\frac{1-1}{1-1} \neq 1,\ \sqrt{4(-1)} \neq 2\sqrt{(-1)} \text{ and } \cos(90°) \times \tan(90°) \neq \sin(90°).$$

Indeed, in each case[5], at least one of the two expressions is undefined.

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[3] Incidentally, we could agree that these equalities refer not to a usual set of numbers, but to the integers modulus two. In that case, they would be verified because the rule $x^2 = x$ holds true in this context. Then, we would have: $(x+1)^2 = x+1 = x^2+1$.

[4] The term *semantic* is used here in its mathematical sense, which opposes syntax (the notations used) and semantics (the objects referred to by those notations).

[5] For the second case, we assume that we are working with the real numbers. Also note that we consider that asserting the truth of $f(a) = g(a)$ presupposes that $f(a)$ and $g(a)$ are both defined.

Note that, in the usual practice of symbol manipulation, rules corresponding to these three "equivalences" are used, sometimes subject to certain "precautions". We intend that this practice be reflected in our theory, which brings us to look for a definition of equivalence with a more general reach.

Semantic Definition of Equivalence, Version 2:
$f(x) \equiv g(x)$ if and only if

- For every element a in $\mathcal{E}$**,** we have: $f(a)$ is defined iff $g(a)$ is defined;
- For every element a in $\mathcal{E}$ for which $f(a)$ and $g(a)$ are defined, we have $f(a) = g(a)$.

We see at once that this new definition settles the case $\sqrt{4x} \equiv 2\sqrt{x}$, but not the cases of $\frac{x-1}{x-1} \equiv 1$ and of $\cos(x) \times \tan(x) \equiv \sin(x)$. In fact, for these last two cases, the right-hand expression is everywhere defined, but not the left-hand one. We hope to improve the situation by proposing a new definition.

Semantic Definition of Equivalence, Version 3:
$f(x) \equiv g(x)$ iff for every element a of $\mathcal{E}$ for which $f(a)$ and $g(a)$ are both defined, we have $f(a) = g(a)$.

One can easily check that the three above-mentioned examples are in fact equivalences according to this new definition. But we still have some problems: Contrary to the preceding definitions, this new definition leads to a non-transitive equivalence, as shown by the following example:

$$|x| \equiv \left(\sqrt{x}\right)^2 \text{ and } \left(\sqrt{x}\right)^2 \equiv x, \text{ but it is not the case that } |x| \equiv x.$$

In fact, in this last example, the first two equivalences are verified (because both sides are defined and equal over the non-negative numbers), while the third equivalence is not verified (because both sides are defined, but not equal, over the negative numbers).

Why is it so important that equivalence be transitive? One crucial reason is that transitivity constitutes an essential part of proofs by the syntactic approach. Let's simply consider our initial example:

$$\begin{aligned}(x+1)^2 &= (x+1)(x+1) = (x+1)x + (x+1)1 = (x+1)x + x + 1 \\ &= xx + x + x + 1 = x^2 + x + x + 1 = x^2 + 2x + 1.\end{aligned}$$

Each rule used allows us to be certain that each expression is equivalent to the next one. But how can we conclude that the first expression is equivalent to the last one? Precisely because of transitivity!

We next try to formulate a definition conciliating transitivity and the presence of values where expressions are not defined. The aim is to restrict ourselves to a subset $\mathcal{D}$ of $\mathcal{E}$ where both expressions are defined.

Semantic Definition of Equivalence, Version 4:
Let D be a subset of $\mathcal{E}$**.** We will say that $f(x) \equiv_D g(x)$ ("$f(x)$ is equivalent to $g(x)$ on D") iff for every element a in D, $f(a)$ and $g(a)$ *are both defined and equal.*

This new definition satisfies a restricted form of transitivity:

$$f(x) \equiv_A g(x) \text{ and } g(x) \equiv_B h(x) \text{ implies } f(x) \equiv_{A \cap B} h(x).$$

Let's see how this can be used in practice. Just cast a new glance at the preceding example:

We have $|x| \equiv \left(\sqrt{x}\right)^2$ on the positive numbers and $\left(\sqrt{x}\right)^2 \equiv x$ on the positive numbers; thus we shall have $|x| \equiv x$ on the positive numbers.

Let's now look at another example:

We have $\dfrac{x-1}{x-1} \equiv 1$ everywhere except in 1,

and also $1 \equiv \dfrac{x-2}{x-2}$ everywhere except in 2.

Thus we will have $\dfrac{x-1}{x-1} \equiv \dfrac{x-2}{x-2}$ everywhere except in 1 and 2.

Note that we used, in this last example, a variant of Version 4 of our definition, where the emphasis is put not on a set $\mathcal{D}$ of numbers where everything is fine, but rather on a set $\mathcal{N}$ of numbers where problems (i.e., restrictions) are present. One can imagine, in the following version, that $\mathcal{N} = \mathcal{E} \setminus \mathcal{D}$.

Semantic Definition of Equivalence, Version 5:
Let $\mathcal{N}$ be a subset of $\mathcal{E}$**.** We will say that $f(x) \equiv g(x)$ except on $\mathcal{N}$ ("$f(x)$ is equivalent to $g(x)$ except on $\mathcal{N}$") iff for every element a in $\mathcal{E}$ but not in $\mathcal{N}$, $f(a)$ and $g(a)$ *are both defined and equal.*

Here is the form taken by transitivity with this Version 5 of our definition:
$f(x) \equiv g(x)$ except on A and $g(x) \equiv h(x)$ except on B implies $f(x) \equiv h(x)$ except on $A \cup B$.

Even if set $\mathcal{N}$ could be any subset of $\mathcal{E}$, we would want it to be as small as possible. But this will not always be possible: Sometimes, $\mathcal{N}$ will have to be finite, countable, even co-finite, as shown by the following examples:

- $\frac{x-1}{x-1} \equiv \frac{x-2}{x-2}$ except in 1 and 2
- $\cos(x) \times \tan(x) \equiv \sin(x)$ except when $x = 90° + k•180°$, where k is an integer
- $\left(\sqrt{x}\right)^2 \equiv x$ except when x is negative
- $\sqrt{x} \equiv \sqrt{-x}$ except when x is nonzero.

So, Version 5 of our definition seems necessary and best suited in situations where expressions are not polynomials (a case where Version 1 is sufficient) or rational functions (a case where Version 3 is sufficient [6]). But we must acknowledge that it seems rather too complex for secondary school students and thus may need to be somewhat transposed for the school environment.

Here are a few suggestions that seem reasonable:

- In lieu of using Version 5 of our definition of equivalence, one could opt, when necessary, for saying "$f(x) = g(x)$ except for the following numbers."
- When applying a sequence of rules, students should be encouraged to refer to the semantics of the situation, using phrases like "without exception", "for all numbers", "for all numbers except …", "except for …", etc., to *qualify* their equalities.
- When using one form or another of transitivity, accumulate all exceptions from all preceding steps.
- If one wanted to use the term "equivalent", one could opt for using Version 3 of our definition, which seems to correspond best with current usage for the secondary level of schooling. In so adopting Version 3, one would eventually have to carry out supplementary verifications so as to "erase" certain exceptions obtained during the sequence of symbolic manipulation.

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[6] In fact, if $f(x)$ and $g(x)$ are quotients of polynomials that take the same values for an infinite number of elements of $\mathcal{E}$, then they take the same values for every element in the intersection of their domains.

Let's apply the preceding suggestions to a simple, albeit somewhat artificial, example[7]:

$$
\begin{aligned}
1 &= \frac{x}{x} && \text{except in } 0 \\
&= \frac{x-1}{x-1} && \text{except in 0 and 1} \\
&= \frac{x-2}{x-2} && \text{except in 1 and 2.}
\end{aligned}
$$

To apply transitivity, we must accumulate exceptions. We can thus conclude that

$$1 = \frac{x-2}{x-2} \qquad \text{except in 0, 1, and 2.}$$

We can verify directly[8] (by substituting values) that this equality also holds when x is 0 or 1. Thus, we will finally have:

$$1 = \frac{x-2}{x-2} \quad \text{except in 2, that is} \quad 1 \equiv \frac{x-2}{x-2} \quad \text{(by Version 3 of our definition).}$$

Now let's apply the above suggestions to a more commonly-encountered situation. Suppose that we want to solve the equation $\cos^2(x) \bullet \tan(x) = 0$. A possible approach is to replace the left-hand expression by a simpler "equivalent" expression, as follows:

$$
\begin{aligned}
\cos^2(x) \bullet \tan(x) &= \cos^2(x)\frac{\sin(x)}{\cos(x)} && \text{except when } \cos(x) = 0 \\
&= \cos(x) \bullet \sin(x) && \text{except when } \cos(x) = 0.
\end{aligned}
$$

Note that, in this example, exceptions are described by a characteristic property: $\left(\cos(x) = 0\right)$, and not by an explicit description: $\left(x \text{ of the form } 90° + k180°, \text{ where } k \text{ is an integer}\right)$.

The problem is thus reduced to a simpler equation

$$\cos(x) \bullet \sin(x) = 0 \ \text{ except when } \cos(x) = 0$$

whose solutions are the union of the solutions of the two following equations

$$\cos(x) = 0 \ \text{ except when } \cos(x) = 0$$

$$\sin(x) = 0 \ \text{ except when } \cos(x) = 0.$$

The first equation is clearly without solutions, while the solutions of the second are simply those of $\sin(x) = 0$, because the sine and cosine functions have no zeros in common.

[7] Please note that the mentioned exceptions apply only to the equality located on the same line; they are not cumulative.

[8] As mentioned in a previous note, this verification is not necessary because we are in the presence of two rational functions that coincide on an infinite number of values of their domain. But such a general result will not always be available.

We have just had a glimpse of the consequences of our treatment of equivalence on the solving of equations; however, we shall not go any further in this direction. We only wish to add the following two remarks: (i) Even in elementary algebra, when we work with expressions other than quotients of polynomials, the notion of equivalence becomes complex because we have to take into account the singular values of these expressions (i.e., the values for which the expressions are undefined); (ii) In this context, it may be well advised to insist less on the notion of equivalence, and more on the domain of validity (or non validity) of equalities; moreover, this approach has the advantage of reminding students of the numerical foundations of symbolic expressions.