CONCEPTUALIZING THE LEARNING OF ALGEBRAIC TECHNIQUE: ROLE OF TASKS AND TECHNOLOGY

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OUTLINE OF THE PRESENTATION

- The first part: The technical-conceptual interface in algebra and what is meant by conceptual (theoretical) understanding of algebraic technique.
- The second part: The ways in which students learn to draw such conceptual aspects from their work with algebraic techniques in a technology environment.
- The third part: A shift to the perspective of teaching practice -- some of the issues involved in planning for the orchestration of task-technique-theory activity in technological environments.

1. INTRODUCTION

- WHAT IS COMPUTER ALGEBRA SYSTEM (CAS) TECHNOLOGY?
  - A **computer algebra system** (**CAS**) is a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form (Wikipedia, Sept. 5, 2007)

CAS USE IN SECONDARY SCHOOL MATHEMATICS CLASSES

- While computers and calculators enabled with symbol-manipulating capabilities have been considered quite appropriate for student use in college-level mathematics courses, and in calculus courses offered at some upper-level high schools (see, e.g., Heid, 1988; Shaw, Jean, & Peck, 1997; Zbiek, 2003), such usage has generally not been the case for secondary school mathematics up to now.

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- Many secondary school mathematics teachers have, in the past, tended to stay away from CAS technology in their classrooms, preferring that their students first develop paper-and-pencil skills in algebra (NCTM, 1999).
- However, these attitudes are changing --
  - based both on research findings and on the leadership of teachers/mathematics educators (and their impact on curricula and tool decisions made at ministerial levels),
  - as well as on the greater availability of teacher resources for using this technology at the Grade 9, 10, and 11 levels of secondary school.

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SO THEN, WHAT DOES THE RESEARCH HAVE TO SAY?

- CAS technology has been found to encourage the use of general mathematical reasoning processes and to improve student attitude, according to PME research reported during the past five years:
  - *"It allows for generating, testing, and improving conjectures"*
  - *"It allows for developing of awareness and intuition"*
  - *"It leads students to explore their own conjectures"*
  - *"It provides non-judgmental feedback"*
  - *"It develops learner's confidence"*

RESEARCH HAS ALSO FOUND THAT …

- CAS can help develop students' knowledge of algebraic content and skills:
  - Their understanding of equivalence (Ball, Pierce, & Stacey, 2003), parameters and variables (Drijvers, 2003), and literal-symbolic algebraic objects in general, without "leading to the atrophy of by-hand symbolic-manipulation skills or to the slower development of these skills" (Heid et al., 2002).

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BUT, WHAT ELSE DOES THE CAS RESEARCH SAY?

- Since the mid-1990s, in France, when CAS started to make their appearance in secondary school mathematics classes, researchers (Artigue et al., 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of the technical work in algebra learning.

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- However, this emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996).
- From their observations, the research team of Artigue and her colleagues came to think of techniques as a link between tasks and conceptual reflection, in other words, that the learning of techniques was vital to related conceptual thinking.

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Our research group was intrigued by the theoretical notion that algebra learning at the high school level might be conceptualized in terms of a dynamic among Task-Technique-Theory (T-T-T) within technological environments.

- And so it came to be that we began a series of studies in 2002, which continue to this day, that explored the relations among task, technique, and theory in the algebra learning (and teaching) of Year 10 students (15-16 years of age) in CAS environments.

In brief, we have found that:

- As reported in Kieran & Drijvers, 2006:
  - Technique and theory emerged in mutual interaction: Techniques gave rise to theoretical thinking; and the other way around, theoretical reflections led students to develop and use techniques.
- As reported in Kieran & Damboise, 2007:
  - A comparative study of a CAS class and non-CAS class revealed that the CAS class improved much more than the non-CAS class in both technique and theory, but especially in theory -- and the sequence of lessons was one where the technical component was clearly in the forefront.

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This brings us to the main question to be addressed in this talk:

- How does the learning of algebraic technique in a CAS environment lead to the emergence of students' theoretical/conceptual growth?
- In other words, how is technique rendered conceptual? What does it mean to have a conceptual understanding of algebraic technique?

2. The interface between technique and theory in algebra

- Note that I will be using the terms *conceptual* and *theoretical* interchangeably.
- Note also that the context of this presentation is related to the letter-symbolic aspects of algebra. Why?
  - A great deal of research exists already with respect to the benefits of multi-representational approaches (e.g., graphical representations) in making algebraic objects more meaningful to students.
  - However, algebra involves more than representational activity; symbolic transformational activity lies at its core.

2. The interface between technique and theory in algebra

- What is meant by a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE*?

  We propose that it includes:
  - Being able to see a certain **form** in algebraic expressions and equations, such as a linear or quadratic form;
  - Being able to see **relationships**, such as the equivalence between factored and expanded expressions;
  - Being able to see through algebraic transformations (the technical aspect) to the underlying changes in form of the algebraic object and being able to explain/justify these changes.

Some classic examples of conceptual understandings in algebra include:

- The distinctions
  - between variables and parameters,
  - between identities and equations,
  - between mathematical variables and programming variables, ...
- Both the knowledge of the objects to which the algebraic language refers (generally numbers and the operations on them) and the need to include certain semantic aspects of the mathematical context so as to be able to interpret the objects being treated. ...

But what might be some examples of that which is intended by '*CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE*'

- 1. Seeing through symbols to the underlying forms, e.g.,

  (a) seeing $x^6 - 1$ as $((x^3)^2 - 1)$

  and as $((x^2)^3 - 1)$,

  and so being able to factor it in 2 ways.

  (b) seeing that x^2+5x+6 and x^4+7x^2+10

  are both of the form ax^2+bx+c.

Examples of what is intended by a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE ...*

- 2. Conceptualizing the equivalence of the factored and expanded forms of algebraic expressions,

  e.g., awareness that the same numerical substitution (not a restricted value) in each step of the transformation process of expanding will yield the same value:

  $(x+1)(x+2)$ -- factored form --

  $= x(x+2) + 1(x+2)$

  $= x^2 + 2x + x + 2$

  $= x^2 + 3x + 2$ -- expanded form --

  and so substituting, say 3, into all four expressions is seen to yield 20 for each exp.

2. The interface between technique and theory in algebra

Examples of what is intended by a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE ...*

- 3. Coordinating the “nature” of equation solution(s) with the equivalence relation between the two expressions that comprise the original equation, e.g., for the following task,

  Given the 3 expressions

  $x(x^2-9), (x+3)(x^2-3x)-3x-3, (x^2-3x)(x+3)$,

  (a) determine which of these three expressions are equivalent;

  (b) construct an equation using one pair of expressions that are not equivalent, and find its solution;

  (c) construct an equation from another pair of expressions that are not equivalent and, by logical reasoning only, determine its solution.

Exp1: $x(x^2-9)$

Exp2: $(x+3)(x^2-3x)-3x-3$

Expr3: $(x^2-3x)(x+3)$

- Which are equivalent?

  Only Exp1 and Exp3 are equivalent.

- An equation using a pair of non-equivalent expressions? And its solution?

  say, Exp1=Exp2

  solution: x=-1(with CAS)

- An equation from another pair of non-equivalent expressions? And its solution?

  Exp3=Exp2; the solution has to be the same as above. Why?

  (a conceptual understanding allows one to answer this last question)

Is it important to foster a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE?*

- National and international mathematics assessments during the 1980s and 1990s reported that secondary school students, in order to cover their lack of understanding, resorted to memorizing rules and procedures and that students eventually came to believe that this activity represented the essence of algebra (e.g., Brown et al., 1988).
- While more recent reform movements have led to infusing “real-world” problem-solving activities into algebra curricula, the traditional dichotomy of skills/procedures and concepts has tended to remain in algebraic discourse.
- Although Skemp (1976) described “relational understanding” as knowing both the rules and why they work, there has never been much movement in the direction of describing what this might mean for algebra.

The role of tasks in the TTT triad ...

- At a recent PME Research Forum on "The Significance of Task Design in Mathematics Education", Ainley and Pratt (2005) -- the organizers of the Forum -- argued that,
- "We see task design as a crucial element of the learning environment ... [and contend that] the nature of the task influences the activity of students."

Also, with respect to tasks:

- Lagrange (1999) suggested that task situations ought to be created in such a way as to “bring about a better comprehension of mathematical content” (p. 63) via the progressive acquisition of techniques in the achievement of a solution to the task.
- Guin and Trouche (1999) added that tasks should aim at fostering experimental work (investigation and anticipation).

- So, to sum up, before moving on:

  With recent advances in

  a) the development of theoretical frameworks, such as that of **Task-Technique-Theory**,

  b) the increasing use of technology in schools, for example, **CAS** at the secondary school level, and

  c) the attention being paid to the role that the nature of the task/situation plays in student learning,

  we are well poised to make headway in reflecting upon the ways in which technique can be viewed from a conceptual angle in the teaching and learning of algebra and, in fact, how technology can enhance such conceptualizing of technique.

3. How Year 10 students in our project drew conceptual aspects from their work with algebraic techniques in a CAS environment

- Concerning the tasks:
  - The tasks went beyond merely asking technique-oriented questions;
  - The tasks also called upon general mathematical processes that included:
    - observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying.
- Concerning the technologies:
  - Both CAS and paper-and-pencil were used, often with requests to coordinate the two;
  - The CAS provided the data upon which students formulated conjectures and arrived at provisional conclusions.

- CONCEPTUALIZING THAT EMERGED WHILE LEARNING NEW TECHNIQUES WITH THE AID OF CAS TECHNOLOGY:

(an example from Kieran & Drijvers, 2006)

  - The task involved factoring (adapted from Mounier & Aldon, 1996).
  - The family of expressions: $x^n - 1$
  - Aim: to arrive at a general form of factorization for $x^n - 1$ and then to relate this to the complete factorization of particular cases for integer values of n from 2 to 13. Proving one of these cases was part of the two-lesson sequence, but is not included today.

One of the initial tasks of the activity

1. Perform the indicated operations: $(x-1)(x+1)$; $(x-1)(x^2+x+1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product
   $(x-1)\left(x^3+x^2+x+1\right) =$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ?
   $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)\left(x^3+x^2+x+1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. Is your explanation valid for the following equality:
   $(x-1)(x^{134}+x^{133}+x^{132}+\ldots+x^2+x+1) = x^{135}-1$? Explain.

After students had worked on these questions, either in groups or individually, the teacher opened up a whole class discussion and asked students to state their responses to one particular question.

- A perusal of the answers offered suggests that different students noticed different things in the pattern of expressions. The teacher's aim in having the whole class discussion was to encourage students to learn from what some of their peers had noticed. Here are some samples of their responses to the given question.

What to focus on, what to notice?

This student noticed that the exponents were different in the second brackets.

2. (c) What do the following three expressions have in common? And, also, how do they differ?

$$(x-1)(x+1),\ (x-1)(x^2+x+1) \text{ and } (x-1)(x^3+x^2+x+1).$$

$(x-1)(x+1)$ $= x^2-1$

$(x-1)(x^2+x+1)$ $= x^3-1$

$(x-1)(x^3+x^2+x+1)$ x^4-1

The $x-1$ is the same in the 1st bracket

x^y y is different

This pair of students focused on the “x+1” that was present at the end of each of the second brackets.

2 c) left bracket = $(x-1)$

End of Right bracket = $\ldots x+1)$

Prior to end of right = increasing x to the power of ...

This student's contribution to the whole class discussion helped others to "refine their noticing."

2. (c) What do the following three expressions have in common? And, also, how do they differ?

$$(x-1)(x+1),\ (x-1)(x^2+x+1) \text{ and } (x-1)(x^3+x^2+x+1).$$

They are all multiplied by $(x-1)$, but each of them adds on an x with a higher exponent in the second equation $((x+1) \to (x^2+x+1) \to (x^3+x^2+x+1))$

After arriving at a general form of factorization for x^n-1 based on a few examples, $x^n - 1 = (x-1)(x^{n-1}+x^{n-2} + \ldots x+1)$, the students worked on the following task for *n* being the integers from 2 to 6, where they were confronted with the completely factored forms produced by the CAS.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

An example of a student's work -- first with p/p (in 1st column), then with CAS (in 2nd column), and then involving a reconciliation of the two (in 3rd column) for x^4-1.

This example shows reconciliation by multiplying the 2nd and 3rd CAS factors:

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

After completing the factorization task for $n = 2$ to 6 in $x^n - 1$, students were presented with the following Conjecture task.

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain.

The following pair of students incorrectly conjectured that, for all odd n's, the complete factorization of x^n-1 would contain exactly two factors.

- Note in the following transcript excerpts how the CAS played a pivotal role in allowing them not only to test their conjecture, but also to refine successively that conjecture.

The moment of surprise when their initial conjecture proves false!

Chris	'Two factors' means two separate sets of brackets, right?
Peter	Yeah.
Chris	The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can't be broken down anymore. 'Cause it always ends up being all positive. And uh, then, because, it's sort of hard to explain.
Peter	When the exponent is [pause], when the exponent is an even number you'll have more than two factors, but when the exponent is not an even number, you'll have exactly two factors all the time.
Chris	Yeah. [Types Factor $(x^7 - 1)$ into the CAS]
	Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] and [Types in Factor $(x^9 - 1)$; the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$]
	And, no!!! [a look of utter surprise on Chris's face]

They wonder: If it is not the case that all odd *n*'s produce exactly two factors when x^n-1 is completely factored, then which *n*'s will produce this?

Chris Hmm, it actually, at a certain point finds uh, that it can be factored more. [types into calculator Factor(x^11–1); The calculator displayed: $(x-1)(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1)$]
Uh, the only case there, x to the nine, x to the eleven, didn't work. Uh [pause], I think there is like occasional, uh, exceptions to the rule [Types into calculator Factor(x^13–1); The calculator displayed $(x-1)(x^{12}+x^{11}+\ldots x^3+x^2+x+1)$]
Like, of what we were saying about it being, uh [Types into calculator Factor(x^15–1)]
we said that all, odd, all times that the exponent was odd it would only have two, but that wasn't true [looks at calculator; the calculator displayed: Factor(x^15-1) equals $(x-1)(x^2+x+1)(x^4+x^3+x^2+x+1)\ldots$]
In a few cases now, because uh, it finds what it simplifies. So, but, I'm not sure how you would be able to tell what were the restrictions.
Wait, if x to the power of twenty-one works, then it may be, uh. [Types into calculator Factor(x^21–1); The calculator displayed: $(x-1)(x^2+x+1)(x^6+x^5+x^4+x^3+x^2+x+1\ldots$]
I think, right now, anytime, it's uh, the exponent can be divisible by three like, [Types in calculator factor(x^27–1); The calculator displayed: $(x-1)(x^2+x+1)(x^6+x^3+1)(x^{18}+x^9+1)$. He also tried out an extreme case, n=99]
Yeah. [Pause] Because, just say an odd number that is divisible by three.

This led to a first revision of their initial conjecture

II.(A).2. Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain:

i.) when the exponent is an odd number that ~~are~~ is not divisible by three with a few restrictions,

ii.) when the exponent is an even number.

iii.) when the exponent is an even number.

But they had not quite finished with their conjecturing and testing of conjectures with the CAS

Chris [he types into the calculator Factor(x^105-1), followed by Factor(x^25-1), followed by Factor(x^15-1)]
That's a pattern [typed in Factor(x^55-1)]
Five is also a restriction. Three's and five's. I think it's a lot more than that (types in Factor(x^49-1)] … I'm just finding that, I think it's when you get past the basic numbers, one through nine, you start running into problems.

Peter Nine gave you three [factors] … Try sixty; sixty is divisible by a lot [Chris types into the calculator]

Chris Yeah, I think it has to do with how many numbers can go into it. Like, sixty is divisible by one, it's divisible by two, it's divisible by three, it's divisible by four, five, six, ten, but then you are doubling it up.

Peter Yeah.

Chris But, it's just, like uh, [pause] at a certain [pause], **prime numbers**? [pause] So, prime number is twenty-three [he types into the calculator] Yeah, prime numbers, that's it! Prime numbers when it is…

Peter Wait, what about three, five and seven.

Chris Only divisible by itself. Three, five and seven, all work

Peter They are prime numbers.

Chris Yeah, they all work

Peter No, but they don't give you exactly two factors.

Chris Yeah, they do. [types into calculator] That's what I'm doing [pause] three, five, seven

Peter Yeah they do [looks at screen]

Chris Yeah, prime factors. And nine doesn't work because it is not a prime factor. [Peter crosses out the answer that he had written previously and writes: all prime numbers]

The last revision of their conjecture regarding the numbers *n* (i.e., prime numbers) that yield exactly two factors for the factorization of x^n-1:

II.(B).2. On the basis of patterns you observe in the table II.B above, revise (if necessary) your conjecture from Part A. That is, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?
ii) contain more than two factors?
ii) include $(x+1)$ as a factor?

Please explain:

i.) ~~odd numbers (for the~~ exponent) that is ~~not divisible by three and five, seven~~ all prime numbers

ii.). composite numbers.

iii). even numbers.

- From these excerpts of Chris and Peter, we have had a glimpse at the role that CAS technology can play in supporting algebraic conjecture-making and conjecture-refining -- allowing these two students to focus their trials on certain multiples of the exponent, to try out extreme cases, ... in short, to arrive at a new conceptualization of the factors for expressions from a certain family of polynomials.

Further evidence for the emergence of theoretical/conceptual ideas arising from work with CAS techniques was gathered from a study we carried out with two classes of weak algebra students.
(Kieran & Damboise, 2007)

- TASK AND TEST DESIGN:
  - A set of parallel activities was developed -- on factoring and expanding.
  - Tasks were identical except that where one class was to use p/p only, the other class was to use CAS or a combination of CAS and p/p.
  - Some tasks were technique-oriented; others were theory-oriented.
  - A pretest and posttest were also created with some questions being technical and others theoretical.

SOME OF THE TASKS:
from Activity 3 (CAS version)

Activity 3 (CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Use the calculator in completing the table below.

Given trinomial (in "dissected" form)	Factored form using FACTOR	Expanded form using EXPAND
(a) $x^2 + (3+4)x + 3 \bullet 4$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why did the calculator not factor the trinomial expressions of 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

And the non-CAS version of the same task:

Activity 3 (non-CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Complete the table below by following the example at the beginning of the table.

Given trinomial (in "dissected" form)	Factored form	Expanded form
Example: $x^2 + (3+4)x + 3 \bullet 4$	$x^2 + (3+4)x + 3 \bullet 4$ $= x^2 + 3x + 4x + 3 \bullet 4$ $= x(x+3) + 4(x+3)$ $= (x+3)(x+4)$	$x^2 + 7x + 12$
(a) $x^2 + (5+6)x + 5 \bullet 6$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why could you not factor the trinomial expressions in 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

IN THIS STUDY, THE TECHNOLOGY WAS FOUND TO PLAY SEVERAL ROLES IN THE CAS CLASS:

- it provoked discussion;
- it generated exact answers that could be scrutinized for structure and form;
- it helped students to verify their conjectures, as well as their paper-and-pencil responses;
- it motivated the checking of answers; and
- it created a sense of confidence and thus led to increased interest in algebraic activity.

THE FINDING THAT: CAS generated exact answers that could be scrutinized for structure and form

- Of all the roles that the CAS played in this study, this was found to be the most crucial to the success of these weak algebra students.
- It proved to be the main mechanism underlying the evolution in the CAS students' algebraic thinking.
- Ironically, the crucial nature of this role was first made apparent to us by the voicing of frustration by one of the students in the non-CAS class:

One of the students of the non-CAS class remarked when faced with these two questions of the task just seen:

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

"How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our paper-and-pencil factorizations and expansions from Question 1 are right."

- This study analyzed the improvements of two classes of weak algebra students in both *technique* (being able to do) and *theory* (i.e., being able to explain why and to note some structural aspects), in the context of tasks that invited technical and theoretical development.
- At the outset, both the CAS class and the non-CAS class scored at the same levels in a pretest that included technical and theoretical components.
- However, the CAS class improved more than the non-CAS class on both components, but especially on the theoretical component.

We see this finding as being of some interest

- Being able to generate exact answers with the CAS allowed students to examine their CAS work and to see patterns among answers that they were sure were correct. This kind of assurance, which led the CAS students to theorize, was found to be lacking in the uniquely paper-and-pencil environment where students made few theoretical observations. The theoretical observations made by CAS students worked hand-in-hand with improving their technical ability.
- In other words, their **technique had become theorized,** which in turn led to further improvement in technique.

4. THE ROLE OF THE TEACHER

- Are good tasks and
  CAS technology
  all that are needed to render technique conceptual, that is,
  to develop a conceptual understanding of algebraic technique?

It would seem not!

- Another crucial ingredient is the teacher's orchestration of classroom activity that gives rise to the conceptualizing of technique in technology environments

Characteristics of teachers' classroom practice involving CAS technology that relate to drawing out the conceptual aspects of technical work in algebra:

- Importance accorded to the mathematical aspects of the task -- both technical and conceptual;
- Emphasis on the mathematical-technological similarities/differences;
- Interest in inquiring into the students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications.

Characteristics of teachers' classroom practice involving CAS technology that relate to drawing out the conceptual aspects of technical work in algebra: --

- Awareness of the many possible roles that the technology can play,

  for example,
    - Create surprising results
    - Generate results for the purpose of exploration
    - Verify other results or conjectures
    - Serve as a computational assistant

- and being able to capitalize on these in such a way as to encourage student learning.

Characteristics of teachers' classroom practice involving CAS technology that relate to drawing out the conceptual aspects of technical work in algebra: --

- Having a repertoire of tasks that engage a variety of learning approaches and evoke different processes, such as
  - Provoking cognitive conflict and seeking to resolve the conflict
  - Looking for patterns
  - Generalizing
  - Activating general mathematical processes, such as observing, comparing, extrapolating, conjecturing, predicting, ...
- And having considered, before the lesson begins, possible student responses and how to encourage further evolution of their thinking within the ensuing lesson.

Characteristics of teachers' classroom practice involving CAS technology that relate to drawing out the conceptual aspects of technical work in algebra: --

- Consideration of ways to incorporate additional artifacts and the roles they might play:
  - This includes worksheets, paper and pencil, blackboard (or the equivalent), …
  - The roles of these other artifacts include:
    - Guiding the work of the pupils and structuring their explorations (worksheets)
    - Focusing the attention of students (blackboard)
    - Leading to a convergence of ideas (blackboard)

Effective teaching practice with CAS would appear to include planning that takes into account at the very least the following:

1. Starting with a key mathematical idea.
2. Thinking about both the technical and theoretical aspects of the key idea.
3. Trying out some technical examples on the CAS to see how best to take advantage of the technology (does it produce any surprises that could be integrated into an interesting sequence?)
4. Deciding what role the technological artifact will play (generate examples, create surprises, serve as calculation assistant, ...)
5. Deciding on the epistemological processes to be engaged (pattern matching and generalization, conjecturing, seeking connections between representations, resolving cognitive conflict, predicting, ...)
6. Reflecting on how to draw out effectively within class discussions the mathematical-technological links.

However, our research so far suggests that the one aspect of teacher's practice in CAS environments that seems to be key to students' becoming aware of the conceptual aspects of their technical work in algebra is:

- Orchestrating classroom discussion in such a way as to draw out students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications.

- This particular discursive aspect of teacher orchestration, within CAS **technology** environments, and in the context of
- **tasks** that
  - go beyond merely asking technique-oriented questions and which
  - call upon mathematical processes that include: observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying, and which
  - require at times that students coordinate CAS techniques with paper-and-pencil techniques, in addition to
  - seeking consistency between surprising CAS outputs and existing theoretical notions,
- has been found to be pivotal to making algebraic techniques more meaningful.

Thank you

Muchas gracias

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