ROLE OF TASK AND TECHNOLOGY IN PROVOKING TEACHER CHANGE: A CASE OF PROOFS AND PROVING IN HIGH SCHOOL ALGEBRA

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Abstract: This article focuses on the teaching practice of one 10th grade teacher, who participated in a research project involving the use of novel tasks and CAS technology. The classroom lessons that are analyzed centered on a proving problem that was embedded within an extended task situation on factoring polynomials. A two-fold analysis is presented, the first one focusing on what transpired during the proving activity. The second analysis draws on and connects the classroom observations with a follow-up interview that was held with the teacher. His reflections during the interview allow us to discuss both what changed in this teacher and what enabled these changes.

Keywords: teachers' learning, teacher change, teaching practice, teachers participating in research projects, proving in high school algebra, tasks, CAS technology

In their review of the emerging field of research in mathematics teacher education, Adler, Ball, Krainer, Lin, and Novotna (2005) have argued that we need to better understand how teachers learn, from what opportunities, and under what conditions. The research findings that we recount in this article provide a compelling case for the particular opportunities and conditions under which the knowledge and teaching practice of a mathematics teacher evolved.

THE CONTEXT OF THE PRESENT STUDY

When our research group[1] developed the program of research that included the present study, it was decided that the use of new technologies (i.e., Computer Algebra Systems – CAS) for the teaching of algebra would be one of its principal components. Another was the design of novel tasks that would both take advantage of the technology to further the growth of algebraic reasoning and focus on the interplay between algebraic theory and technique. The theoretical framework that underpins the research, one that we refer to as the *Task-Technique-Theory* frame (see Kieran & Drijvers, 2006, for details), draws upon Artigue's (2002) and Lagrange's (2002) adaptation of Chevallard's (1999)

[1] Our appreciation to the other members of the research group, which included at the time that the program of research was elaborated, André Boileau, Denis Tanguay, Fernando Hitt, and Luis Saldanha, as well as the consultant, Michèle Artigue.

anthropological theory of didactics. From their research observations, Artigue and her colleagues came to see techniques as a link between tasks and theoretical reflection, in other words, that the learning of techniques was vital to related theoretical thinking. Based on this notion, our research group developed a research program that conceptualized algebra learning at the high school level in terms of a dynamic among task, technique, and theory, within technological environments.

At the same time that we began to create a series of tasks that would invite both technical and theoretical development in 10th grade algebra students, we also made contact with several practicing mathematics teachers to see if they might be interested in collaborating with us. The form of collaboration that we arranged was on several levels. First, the teachers were our practitioner-experts who, within a workshop setting, provided us with feedback regarding the nature of the tasks that we were conceptualizing. Second, after modifying the tasks in the light of the teachers' feedback, we requested that, at the beginning of the following semester, they integrate the entire set of tasks into their regular mathematics teaching and that they be willing to have us act as observers in their classrooms. Third, throughout the course of our classroom observations, which occurred over a five-month period in each class, we also offered a form of ongoing support to the participating teachers by being available to discuss with them whatever concerns they might have. In addition, we conducted interviews with some of them immediately after certain lessons that we had perceived to be worthy of further conversation, lessons that we had thought might even be considered pivotal moments in their practice. The following narrative concerns one such pivotal two-lesson sequence, taught by the teacher Michael.

MICHAEL'S STORY

Some Background

Michael was one of the teachers involved in the project. Up to the time of the present study, we had already observed 15 of his classes, that is to say each of the lessons in which he had thus far integrated a CAS-supported task from the set that had been created for the research project. Michael, whose undergraduate degree and teacher training had been done in the U.K., had been teaching mathematics for five years, but he had not had a great deal of prior experience with technology use in mathematics teaching, except for the graphing calculator. He was a teacher who, along with encouraging his pupils to talk about their mathematics in class, thought that it was important for them to struggle a little with mathematical tasks. He liked to take the time needed to elicit students' thinking, rather than quickly give them the answers.

We began to observe Michael's class from the very beginning of the Grade 10 school year. The students in this class had learned a few basic techniques of factoring polynomials (for the difference of squares and for factorable trinomials) and the solving of linear and quadratic equations during their 9th grade mathematics course. They had used graphing calculators on a regular basis; however, they had not had any experience with symbol-manipulating calculators prior to the onset of our project, which made use of the *TI-92 Plus* hand-held, CAS calculator. These students were already quite skilled in algebraic manipulation, as was borne out by the results of a pretest we administered at the beginning of the study; but we were informed that they had never engaged in any activity related to proving, either in geometry or in algebra.

This article concerns the two lessons that had involved the $x^n - 1$ task set (hereinafter referred to simply as the $x^n - 1$ task), the last component of which was a proof problem. We observed, and videotaped, both of these class lessons. The day after the close of the two lessons, the first author interviewed Michael. The next few paragraphs describe first the task and then our classroom observations of the proving segment of the task, followed by an analysis of this activity. Then we present extracts from the interview with Michael and a second analysis that draws on both his interview reflections and our earlier classroom observations.

The x^n - 1 Task

The design for the two-lesson sequence was an elaboration of earlier work carried out by Mounier and Aldon (1996) with their 16- to 18-year-old students on a task that involved conjecturing and proving general factorizations of $x^n - 1$. Our task activity had three parts. The first part, which involved CAS as well as paper and pencil, aimed at promoting an awareness of the presence of the factor $(x-1)$ in the given factored forms of the expressions $x^2 - 1$, $x^3 - 1$, and $x^4 - 1$ (see Figure 1), as well as leading to the *generalized* form $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + x + 1)$.

1. Perform the indicated operations: $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product $(x-1)\left(x^3 + x^2 + x + 1\right) =$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ? $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x-1)\left(x^3 + x^2 + x + 1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. On the basis of the expressions we have found so far, predict a factorization of the expression $x^5 - 1$.

Figure 1. Some of the initial tasks of the activity

The next part of the activity involved students' *confronting* the paper-and-pencil factorizations that they had produced for $x^n - 1$, with integer values of *n* from 2 to 6 (and then from 7 to 13), with the completely factored forms produced by the CAS, and in *reconciling* these two factorizations (see Figure 2).

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 2. Task in which students confront the completely factored forms produced by the CAS

An important aspect of this part of the activity involved reflecting and *forming conjectures* (see Figure 3) on the relations between particular expressions of the $x^n - 1$ family and their completely factored forms.

Conjecture, in general, for what numbers *n* will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x+1)$ as a factor?
Please explain.

Figure 3. Task in which students examine more closely the nature of the factors produced by the CAS

The final part of the activity (see Figure 4) focused on students' *proving* one of the conjectures that they had generated during the previous part of the task. This proving activity is the central component of the analysis of teacher practice and teacher change that we present in this article.

Prove that $(x+1)$ is always a factor of $x^n - 1$ for even values of *n*.

Figure 4. The proving task

Our Classroom Observations

After students had completed the first two parts of the $x^n - 1$ activity, they were faced with the proving segment of the task: Prove that $(x+1)$ is always a factor of $x^n - 1$ for even values of *n*. Mathematically experienced students might possibly have been able to generate a proof along the following lines:

$$x^n - 1 = x^{2k} - 1 \qquad \text{(for } n \text{ even)}$$

$$= (x^2)^k - 1$$

$$= (x^2 - 1)(x^{2^{k-1}} + x^{2^{k-2}} \dots + 1)$$

$$= (x + 1)(x - 1)(\quad \dots \quad)$$

However, our research team did not consider for an instant that such a symbolic form of proof might be forthcoming from the 15- and 16-year-olds in our study. Nevertheless, we did entertain the idea that some generic form of proof might be produced. For example, students might propose that the expression x^{18} (where the 18 represents any even integer) could be viewed as $(x^2)^9$, and thus that $(x^{18} - 1)$, which is equivalent to $((x^2)^9 - 1)$, could be factored according to the general rule for $(x^9 - 1)$, but with the x being replaced by x^2. As mentioned earlier, the students of Michael's class had not had any prior experience with proving in algebra. Such lack of experience with proving is not unusual for students of this age. This is reflected in the general absence of algebraic proving activity among high school students in the research literature. Nevertheless, some attention has been given to number-theoretic proofs (e.g., Healy & Hoyles, 2000; see also Mariotti, 2006), as well as to proofs involving geometric figures (Balacheff, 1988). However, we could find nothing that was closely related to algebraic proofs of the kind being proposed within our $x^n - 1$ task. Mounier and Aldon's (1996) report, which had stated that students generated four proofs for various factorizations of $x^n - 1$ where n is a positive integer, did not describe the actual activity of proving nor provide the steps of the students' proofs.

To return now to our observations of the unfolding of the proving activity in Michael's class, the students worked on this part of the task, mostly within small groups, for about 15 minutes. Some were using their CAS calculators, but most were just talking about how they might approach the task and occasionally jotting things down on paper. During that time, the teacher circulated and was heard to offer the following remarks to various groups (T = Teacher):

T: See if you can prove this and not just state it, as some people have done so far (picking up one student's worksheet and reading it to the class): 'When n is greater than or equal to 2, $(x+1)$ is a factor because.' Let's see if we can go a little bit beyond that. Can you write down what you come up with. ... Yeah, but you need more than just examples. … You need to get something written down. … Look, you need to think in order to answer this. This is the only hint I'm giving you, you need to think about where the $(x+1)$ comes from.

Getting students into this proving task was not straightforward, as they had never before engaged in such mathematical activity. However, with the teacher's encouragement, they did make progress. When he sensed that the majority of them had arrived at some form of a proof, he opened up a whole-class discussion, oriented around various students' sharing their work:

T: Ok, guys. Quite a lot of you got quite close in doing this. What I want you to do, and I've asked a couple of people who've done it in completely different ways, to see if they can put forward their explanation. I want you to be quiet, listen to their explanation, then we'll discuss it once they've got it done, once they've completed their little spiel, ok.

He invited selected students to come to the board, one at a time. As will be seen, the principal contributions of the students can be grouped into three distinct approaches. The first proof, which is presented immediately below, revolves around the idea of 'difference of squares'. Despite a follow-up counterexample involving the 'sum of cubes', and a return to the validity of the notion of 'difference of squares', the proof-giver never quite fills in the gaps to arrive at a full proof.

Proof 1: A general approach based on the difference of squares. Paul was invited to come to the front and to present his 'proof':

Paul: Ok. So, my theory is that *whenever x^n-1 has an even value for n*, if it's greater or equal to 2, that, *one of the factors of that would be x^2-1*, and since x^2-1 is always a factor of one of those, *a factor of x^2-1 is (x+1), so then (x+1) is always a factor*.
S2: Could you say it again? [other students react all at once, making many comments]
S3: Why don't you write it on the board?
T: Guys! Give him a chance.
Paul: You want me to write? [addressing the teacher]
T: Write down what you want to write down.
S4: Can you talk at the same time?

Paul then proceeded to write down at the board that which he had just stated orally. The teacher then asked: "Is everyone willing to accept his explanation?" While many seemed to agree with what Paul had proposed, a few voiced disagreement – to which the teacher responded: "Ok, guys, one at a time. Ok, start with Dan."

A proposed counterexample involving the sum of cubes. Dan then came forward with what he considered a counterexample, $x^{12}-1$, to Paul's proof. Dan proceeded by factoring $x^{12}-1$ as $(x^6+1)(x^6-1)$, the latter of which he refactored as $(x^3+1)(x^3-1)$. His subsequent factoring of (x^3+1) – a sum of cubes – yielded the sought-for (x+1) factor (see Figure 5). Thus, he maintained that the presence of x^2-1 was not necessary for a proof because he (Dan) had shown that, for even values of *n,* the factoring of x^n – 1 does not have to end up with a difference of squares. A sum of cubes could result, and it too would yield a factor of (x+1). This led immediately to many students' voicing disagreement, to which the teacher remarked:

T: Ok, so, so this is good [he points to the third line on the board, which contains $(x^3-1)(x^3+1)(x^6+1)$]. This is good because, Paul, the problem I had with yours, is how do you get from here to here [he points to x^n-1 and then to the x^2-1 of Paul's board work; he then draws a red arrow to highlight the gap between those two lines of the

proof], does that follow? He's just given you a counterexample where it does not follow.
Some Students: It does though [with many students speaking at once].

The counterproposal containing seeds of a generic proof. Many of the other students, including Paul, contended that Dan's was not a counterexample, after all. They argued that the expression $x^{12} - 1$ could, in fact, produce $x^2 - 1$ if it were factored differently:

Paul: Isn't x^6+1 a sum of cubes? … *So couldn't you also do the* x^6*-1 as the difference of cubes* [one student says "yeah"] *and that's* x^2*-1.*
T: [he circles (x^6-1) in red and draws an arrow on the left to show the alternate factorization being proposed. See the leftmost arrow and its accompanying factorization in Figure 5]
Paul: [continuing what he was saying] So x^2-1 times whatever [the teacher writes (x^2-1)(x^4+x^2+1) on the board]. So there's your x^2-1.
S5 (a student other than Paul): Even though it's not fully factored [referring to $x^{12} - 1$], x^2-1 is still a factor of that.
Paul: Sir, it can be factored down
T: Yeah I know it can be factored down, and I am not saying you're wrong, what I'm saying is that your reasoning to get from x^n-1 down to this [he points to the x^2-1 line of Paul's proof] is not complete. Do you agree (to Paul)?

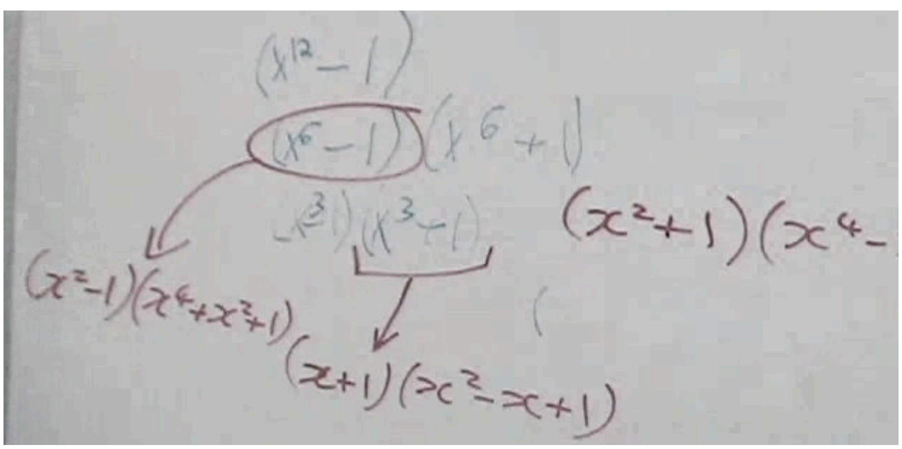

Figure 5. Dan's counterexample in the central section, with the counterproposal by Paul that x^6-1 does indeed yield x^2-1 (at the leftmost arrow)

Analysis of Proof 1. While Paul had seen that $x^6 - 1$ could be viewed as a difference of cubes, and thus that $x^2 - 1$ was a factor, he did not seem able to link this particular example with his general affirmation that for all even *n*s in $x^n - 1$, one would always arrive at $x^2 - 1$ as a factor. Yet, he was unbelievably close. Could he see that $x^6 - 1$ was equivalent to $((x^2)^3 - 1)$, even if he had never expressed it in quite this way? Or was his realization based solely on his experience with factoring the 'difference of cubes' and merely with perceiving 6 as a multiple of 2 and of 3? If the former, why not see also that

$x^8 - 1$ was equivalent to $((x^2)^4 - 1)$, … , and more generally that $x^n - 1$ for even *n*s could be expressed as $((x^2)^p - 1)$ where $n = 2p$? And so if $x^n - 1$ has $(x\text{-}1)$ as its first factor, why not then see that, similarly, $((x^2)^p - 1)$ would have $x^2 - 1$ as its first factor, and thus $(x+1)$ as a factor? While Paul had certainly intuited some of this in offering his initial proof, the connections were likely still quite tentative and not yet able to be formulated in an explicit way. However, the teacher, Michael, had insisted that, for Paul's proof to be complete, there needed to be a theoretical link connecting the two main lines of the proof (the $x^n - 1$ line and the $x^2 - 1$ line): "Yes, we know we will get there eventually, but how do we know that we will eventually get there without doing all the actual factoring?" Paul's proof had a 'gap' in it (see Weber & Alcock, 2005, for more on 'gaps').

Proof 2: A proof involving factoring by grouping. The second approach to the proving problem was put forward by Janet. Janet's proof, which she and her partner Alexandra had together generated, was based on their earlier work on reconciling CAS factors with their paper-and-pencil factoring (for the tasks shown in Figure 2). They had noticed that for even *n*s, the number of terms in the second factor was always even. Janet argued, as she presented the proof at the board using $x^8 - 1$ as an example, that it would work for any even *n*:

Janet: When *n* is an even number

T: Write it on the board, show it on the board.

Janet: [she writes "x^8-1" and below it: $(x\text{-}1)(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$]

T: Ok, listen 'cause this is interesting [addressed to the rest of the class], it's a completely different way of looking at it, to what most of you guys did. Ok, so explain it, Janet.

Janet: *When n is an even number* [she points to the 8 in the x^8-1 that she has written], *the number of terms in this bracket is even, which means they can be grouped and a factor is always (x+1).*

T: Can you show that?

Janet: [she groups the second factor as follows, $x^6(x+1)+x^4(x+1)+x^2(x+1)+1(x+1)$]

T: Thanks Janet. Do we understand what she put out there?

A student's query related to the factor (x+1). As soon as Janet had finished the writing of her proof at the board, another student posed a rather insightful question: "*But how do you know that the group is going to be (x+1)?*" As no student could offer any response to this, the teacher Michael interjected with a general notation for Janet's proof, in the hope that this might perhaps help the questioner to see the logical necessity of the (x+1) factor (It is noted that Michael remarked to us during the classroom observations that Janet's proof was one that he had not thought of before; yet, he was able to react quickly with a general formulation in response to the '(x+1)' question.):

T: You know it's going to be x^{n-1} plus x^{n-2} plus dot, dot, dot, plus x plus 1 [he writes on the board as he speaks] and you know there's an even number here yeah? [he points to the series of dots in the polynomial]. Yes? So you know that, in there, if we take this [he points to the x^{n-2}] as the term outside. You know that these two [he points to x^{n-1} and x^{n-2}] can be factored and it's just (x+1) as the other factor of these two [he wrote $x^{n-2}(x+1)$], yeah? And that would be the case for any and all in between [he points to the series of dots], and including this [he points to the '+ x + 1' at the end of the sequence and writes 1(x+1) on the far right of the x^{n-2}(x+1) that he had already written; see Figure 6].

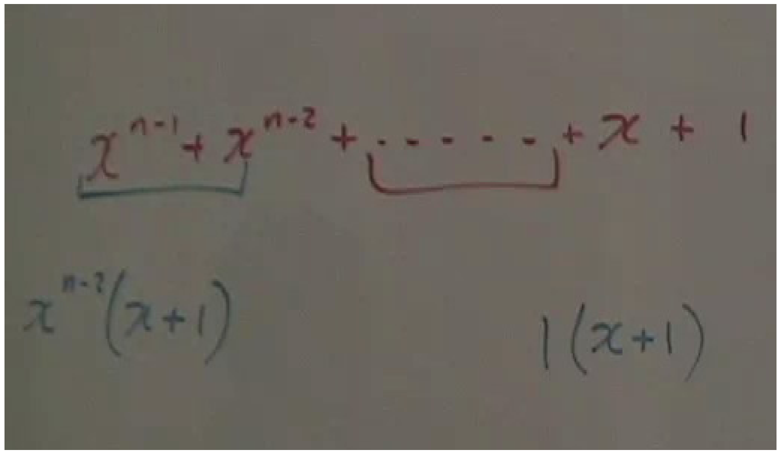

Figure 6. A general notation illustrating that (x+1) will be a factor upon grouping

Analysis of Proof 2. Janet's proof, which was generic in that it embodied the structure of a more general argument and was a representative of all similar objects (Balacheff, 1988; Bergqvist, 2005), was one that seemed to be understood and appreciated by most of the students in the class (see Weber, 2008, for related discussion). It also provided insight as to why the proposition holds true not only for that single instance but for all related cases (Rowland, 2002). Janet had been able to explain how the terms of the second factor (the factor beginning with the x^7 term) could be grouped pair-wise, yielding a common factor of $(x+1)$, even if she did not complete the factoring process:

$$\begin{aligned} x^8 - 1 &= (x-1)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \\ &= (x-1)\left[x^6(x+1) + x^4(x+1) + x^2(x+1) + 1(x+1)\right] \\ &= (x-1)(x+1)(x^6 + x^4 + x^2 + 1) \\ &= (x-1)(x+1)(x^2+1)\left(x^4+1\right) \end{aligned}$$

Janet's proof had appealed to her classmates' common experience in factoring by grouping. But, as has been discussed by Balacheff (1987), generic proofs such as these may often use rather imprecise tools and be defective in certain respects – as was, for example, pointed out by her classmate's question as to how Janet knew that (x+1) would appear in the grouping, or the unposed question as to how she knew that there would always be an even number of terms in the second factor of the first line of her proof. Nevertheless, it was a clever proof with a degree of elegance that indicated to the teacher, Michael, that his students could go much farther in the activity of proving than he had initially thought possible (compare with Bergqvist, 2005, where teachers were reported to believe that only a small number of students can use higher level reasoning).

Proof 3. A new conjecture involving x^n + 1 where n is an odd integer. When Paul had presented his proof to the class, the implicit underlying argument was that when one begins with x^n – 1 where n is an even integer, and if one continually takes the even exponent and treats it as a difference of squares, then one eventually arrives at $x^2 – 1$. Shortly after Janet had finished explaining her proof, the issue of Paul's proof came up once more. To provoke the students, the teacher offered the following counterexample: "Just out of interest, what would happen if this was $x^{14} – 1$? [he wrote ($x^{14} – 1$) under the ($x^n – 1$)], to which a student easily responded: "(x^7-1) times (x^7+1)." The teacher wrote at the board ($x^{14} – 1$) = (x^7-1)(x^7+1) and then wondered aloud: "Where does that leave your proof, Paul?" However, rather than leaving the class stymied, this question provided an opening for another student who had been conjecturing something new:

Andrew: See, when it's a prime number, then the first part here is x+1 as a factor. … From, like x^5+1 you get, x^4-x^3+x^2-x+1, like when you factor it on the calculator, that's what you get.
T: Ok.
Andrew: x+1 times x^4-x^3+x^2-x+1.
T: Say it again Andrew [he is ready to write down Andrew's verbalizings at the board]
Andrew: When you factor x^{10}-1 on the calculator, you get (x-1) times (x+1) times (x^4+x^3+x^2+x+1) times (x^4-x^3+x^2-x+1).
T: Yeah [while completing the writing of Andrew's factorization at the board]. So, just go back a bit. That was these two together [tracing an arc joining (x-1) and (x^4+x^3+x^2+x+1)] to give you the x^5 -1.
Andrew: Yeah, and the next two would be (x+1) and (x^4-x^3+x^2-x+1) [See Figure 7].
T: So you're going into something that we haven't looked at in this class. You're setting up another hypothesis. What is your hypothesis?
Andrew: Well, that's what I was trying to get at. … If the division by 2 gives an odd number, then it goes (x+1).
T: So you're saying that, for the second hypothesis, something like this [he writes down (x^5+1)=(x+1)(x^4-x^3+x^2-x+1), just as the bell rang]. And you're saying that's true for all odd numbers?
Andrew: That's what I think.
T: So if we could prove this, then we've got it. But we've run out of time.

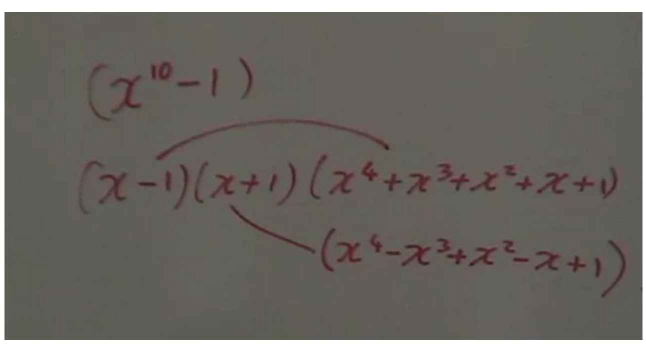

Figure 7. Moving toward a conjecture involving x^n+1 for odd ns

Analysis of Proof 3. When Andrew had been working earlier on the second part of the $x^n – 1$ task, which had involved the reconciling of his paper-and-pencil factoring with the CAS factoring, the $x^{10} - 1$ example had presented a surprise. He had first factored it with pencil and paper as $(x^5+1)(x^5-1)$, and then refactored the (x^5-1) according to the newly-learned general rule, but had left the (x^5+1) factor as is. But the CAS produced as its factored form for $x^{10}-1$: $(x-1)(x+1)(x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1)$. Andrew noticed this additional factoring by the CAS, that is, that $x^5+1 =$ $(x+1)(x^4-x^3+x^2-x+1)$. So, he began to conjecture and test the more general rule:

$$x^n+1=(x+1)(x^{n-1}-x^{n-2}+...-x+1)\text{, when } n \text{ is odd.}$$

It is interesting that, as Andrew was explaining his conjecture to the teacher, it was clearly a new idea for the teacher too. While Andrew's conjectured new rule did not address the gap in Paul's proof, it did provide a worthy response to the teacher's (x^{14} – 1) counterexample: that is, that it did not matter if the 'difference of squares' approach led to exponents that were odd integers, by taking the 'plus' factor (e.g., x^7+1), one would still end up with a factor of (x+1)!

A few remarks regarding the proving part of the activity. Keeping in mind that the proving attempts that we have just witnessed were generated by 15- and 16-year-olds with no prior experience in algebraic-type proofs, their work is indeed remarkable. Hanna (2005) points out that, "While in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation" (p. 47). She adds that, "A good proof, however, must not only be correct and explanatory, it must also take into account, especially in its level of detail, the classroom context and the experience of the students" (p. 48). While the explanatory power of Janet's and Andrew's proofs was in a sense stronger than that of Paul's, even his had the seeds of a powerful explanation.

How might we account for the richness of the students' work with respect to proving? Mariotti (2002) has argued that there is no proof without theory. In the same vein, Mariotti and Balacheff (2008) have emphasized that the proving process as a complete whole necessarily starts with the production of conjectures before moving on to proof. In this regard, it is noted that the entire two-lesson sequence that was devoted to the x^n – 1 task involved an interplay between theory and technique. Such is the backbone of all the task activities developed within the present project. For the given task, the development of student conjectures was requested right from the second question in the first part (see Figure 1). Further conjecturing that was more explicitly related to the proving task was also called for in the second part of the activity (see Figure 3). Thus, the ideas that the students generated during the proving task were those that they had been conjecturing about and playing with throughout the entire activity.

The findings of this study contrast with some of the prior work on proof and justification that has been reported in the literature. For example, Healy and Hoyles (2000) found from their study of 14- and 15-year-olds' conceptions of proving within the number-theoretic domain that students were more likely to prefer empirical to algebraic arguments: "Regarding explanatory power, arguments that incorporated algebra were most likely to be viewed neither as showing why the given statement was true nor as representing an easy way to explain to someone who was unsure" (p. 414), … students were put off from using algebra because it offered them little in the way of explanation;

they were uncomfortable with algebraic arguments and found them hard to follow" (p. 415). These findings of Healy and Hoyles are consistent with the results of Lee and Wheeler (1987) who found that high school students preferred numerical examples to algebraic proofs and did not view algebra as a tool for justification and proof.

So why were the students of our study so impressed with the explanatory power of the algebraic proofs generated by themselves and their peers? There are a couple of major differences between the kind of proving tasks used in the two studies referred to above and the task used in this study, differences that can explain the divergence in the findings. First, as already pointed out above, the $x^n - 1$ task had built into it a great deal of prior conjecturing activity that was related to the ideas that were integral to the proving part of the task. This is in contrast to the proving tasks used in the Healy and Hoyles survey that were not preceded by prior student activity in developing related conceptual ideas. Students of the Healy and Hoyles study were confronted with the survey instrument 'out of the blue', so to speak. Second, and this is a critical difference, the Healy and Hoyles problems presented students with statements, such as, "When you add 2 even numbers, your answer is always even", followed by choices of proof responses that included numerical, algebraic, and pictorial approaches. That students tended to choose numerical justifications as being more convincing is not surprising, given the numerical aspect of the initial problem statement. However, the $x^n - 1$ task is not a task that suggests numerical exemplification as does the above number-theoretic task. Furthermore, most of the tasks in the Healy and Hoyles study were presented to the students in a verbal rather than an algebraic form, the latter of which was the case in our study and which may have induced students to embrace an algebraic form of proof. In fact, the proving activity of the $x^n - 1$ task did not evoke the usual dialectic between the numerical and the algebraic – as is often the case in algebraic activity – but rather a higher-level dialectic between specific algebraic examples (e.g., $x^2 - 1$) and more general algebraic formulations (e.g., $x^n - 1$). The students in the current study remained at an algebraic level throughout – one that had been supported by a great deal of prior work involving related conjecturing.

Hanna and Barbeau (2008) have advanced the notion that, "Proofs yield new mathematical insights, new contextual links and new methods for solving problems, giving them a value far beyond establishing the truth of propositions" (p. 346). We would add, in closing this section and in introducing the next, that these new mathematical insights were found in our study to flow both in the direction of the proof-giver and in the direction of the proof-receiver – the proof-receivers being both students and teacher.

The Subsequent Interview with Michael

The 35-minute interview with Michael took place at the close of the proving activity. It inquired into a range of issues related to his views on the research project, as well as his impressions of the most recent activity involving the $x^n - 1$ task. The main thrust of his reflections, which are captured in the following verbatim extracts, focus first on his initial expectations, then on his changed views after having experienced a few months of classroom activity with the project tasks and CAS technology, and finally on related future plans.

His initial expectations – Extract 1.

Interviewer: Did you have any expectations or apprehensions about the proposed use of symbolic calculators in your math class before this project started?

Michael: Hmh, I guess I wasn't sure how it was going to go; I was apprehensive to some degree. I was a little bit concerned about how the students would take to it and whether they would see it as being dragged away from what they needed to do. I was a little bit worried about how the parents would take it, but that's been no issue at all. In parent-teacher interviews, a lot of them said they were quite pleased that we're doing some of these things and pushing the kids a little bit further. My feeling about the project itself was that we had enough time to do it, so it couldn't be bad. I'd figured you guys had put some thought into what you were doing and there was a good chance that it was going to be successful and to help them a little. I don't know if my expectations were that huge, but I was hoping there would be something there.

His changed views – Extract 2.

Interviewer: Do you now see this technology as playing a different role in your class from the time before the project started?

Michael: Yes, for sure, because before the project started, like I said, I hoped it would be good, but my expectations were not that high about it. I certainly have been very pleasantly surprised with what's happened and I don't think I would have considered when we did this in June last year – when we went for the training days – I don't think I would have considered that I would be at this stage. I didn't think I would have been in a situation where I'd be saying to you: "I want to use this again next year." I don't think that those were my expectations, I thought it would be ok and kind of fun, and a nice diversion, but I didn't think we would be quite at the level that we are. I guess my expectations were a lot lower than what we've achieved.

Brief commentary on Extracts 1 and 2. Michael had not had high expectations at the outset of the project. This makes the results all that much more interesting and persuasive. Some mathematics educators have been heard to express some reservation regarding the role that technology can play in the learning of mathematics, a few even suggesting that it is the already-converted with their 'rose-tinted glasses' who are technology's greatest proponents. Yet, here we have a teacher who was not already 'converted' at the start of the project and who, as the project progressed, became highly impressed with the way in which the technology was serving to enrich the mathematical learning of his students.

The tasks and the technology – Extract 3.

Interviewer: If I were to ask you to describe what you think has been the impact on the students of this project both mathematically speaking and technologically speaking, how would you describe this impact?

Michael: I think the biggest impact, and the thing I've been most happy with is the way you guys have designed the activities. It's the way that we've challenged their [the students'] thinking and actually made them think about a process that maybe they knew how to do, but made them think about why they're doing it that way. And I think that's what the calculator has helped them to do and helped them to really, really look at whether they understand the material – basically the meta-cognition kind of idea of thinking about the process you're going through yourself, the thinking you're going through. That's something we don't do enough of in mathematics, I think we should do and I really like to do it. So in a mathematical sense I would say that's been the biggest thing. … The learning through the technology was amazing. But it's the amount of work that you put into these activities, and that's why they were so successful. The technology is nothing by itself. That's why it's been such a pleasure to do this and why I have really enjoyed it – it's because you people clearly know what you're doing and have spent a lot of time organizing these. Like you said, we [the teachers] were involved, but only to a small extent. And it's been really good to see how the kids have developed with these [the tasks] and worked with them.

Change in his teaching – Extract 4.

Interviewer: Has this project affected your style of teaching in any way?

Michael: [shy-laughing] I don't know. I think it's made me think more, or made me realize that what I like is making them think a little bit more. And I think I did that anyway, I remember when you came into class last year that there were some things similar happening, but it just made me, just consider a little bit more: Can I let them come through this themselves, let them try this out themselves a little bit more, which I think I always did – but just seeing these activities work, it's made me realize there's more scope to it than I have done in previous years. There is much more scope to let them really go and really know the material properly. So, [to answer your question] I think so, a little bit.

Pushing students to go farther mathematically – Extract 5.

Interviewer: Has the project altered your view of the nature of the mathematics content that can be taught at this level?

Michael: Yes. Because some of the things that you had in those activities I wouldn't have touched. Such as, especially the last activity [the $x^n - 1$ task set], you know there's no way I would have gone anywhere with that. It was way beyond anything that they need to know, but just doing that activity was such a fulfilling experience for, not just for me, I spoke with some of the kids afterwards, and they really enjoyed it. They really did! Just going way beyond what they needed to do [in the math program for that grade level] and they were all able to do it. The really nice thing about that activity is that, at the end of it, everyone had something. Even if they didn't all have as nice a little proof as Janet and Alexandra, all of them had worked someway along the lines to get to something. So, so yeah, it certainly opens up things and they couldn't have done that without the technology. So, so for sure is the answer to your question.

Brief commentary on Extracts 3, 4, and 5. While the CAS technology was deemed by Michael to be essential to the changed nature of his students' mathematical learning, he was quick to point to the role played by the task activities. He emphasized that the technology by itself would not have produced that which he and his students experienced in this new learning environment. It was the tasks and the way in which they pushed students beyond what is normally asked of them in their mathematics program that seemed crucial. The task sheets included questions that were not only different, and also rich and challenging extensions of that with which they were already somewhat familiar, but which proved to be quite feasible because of the presence of the CAS technology. In addition, the content of the tasks provided Michael with the grist needed to pose additional questions and to encourage his students to think hard about difficult ideas. The intertwining of novel and substantive mathematical tasks, and technological tools appropriate for these tasks, led to mathematical activity that the students quite enjoyed and from which they learned a great deal. This, in turn, promoted the development of new awarenesses on the part of the teacher, awarenesses that will be discussed shortly.

Using technology to increase student involvement and promote learning – Extract 6.

Michael: With this technology, the learning is not the same [as with my teaching at the board]. Learning goes much further, it is much more involved. That's why I have really, really enjoyed it. Normally, I'd be involving about two or three of them, but not the entire class. With this tool, it gives them the extra level of ability, and it involves more students. It gets them into it a lot more.

Interviewer: You mentioned that the technology in combination with the activities made them think a lot more about their mathematics, and some of the different steps in the process. What if they hadn't had the technology, could you have seen or imagined that the same sort of progress would take place with similar activities but not incorporating technology?

Michael: I guess with some of the activities it would have been possible, but I think with some of them it would have been either impossible or very, very fake 'cause you'd have to give them answers anyway, you'd have to give them results from the calculator. If you take the last activity, the activity 6 [the $x^n - 1$ task] that we did. The only way you could have done that without the technology would have been to give them what the calculator gave them itself. So then it becomes less hands-on, they don't get into it as much. The fact that they derived these things and went through the process of "there it is [gesturing to the right to suggest a paper-and-pencil result], that's what the calculator gave [gesturing to the left – emulating a comparison], how do I reconcile the difference, how do I factor this, how do I do it?" I think that without the technology it would be so artificial that it would lose them. And basically, you would have had to use the technology at some level anyway to give you the answers, so the fact that they could discover things themselves is a valuable effect.

Future plans – Extract 7.

Interviewer: Do you see yourself using this technology and these activities perhaps next year in the same class?

Michael: I was actually saying to David [his colleague and department chair] that I fully intend using them. You know that our school is in the process of moving toward laptops and it's interesting that when you do that, there is a lot of what I would call – to use a derogatory term – 'fluff' to it: "ok, that's a nice powerpoint, that's a nice little internet site to look at this, or you can use a smart board and you can highlight this and that looks really pretty." So, when I asked you for the DVD of the class lessons on the $x^n - 1$ task, the reason why I asked for it is that I want to show people [my colleagues] what can actually be done. Even when we were not actually using the technology all the time [in class], the learning through the technology was huge.

Brief commentary on Extracts 6 and 7. In orchestrating his classroom teaching, Michael did not always use the technological artefact of the classroom view-screen hooked up to a TI-92 CAS calculator. Yet, he was firmly convinced that the "learning through the technology was huge." Not only did it allow the students to go farther mathematically, it encouraged them to be more active and more involved participants in the process of learning. Michael's impressions of the project activity were so favourable, in fact, that he wanted to continue using the tasks and the technology the following year; he also wanted to share the video of the lessons of the last two classes on the $x^n - 1$ task with his colleagues, just so that they could see what is possible with this technology and with the kinds of tasks that were developed within the project.

The above interview-highlights are now analyzed in the light of what we observed during the two classes, with a view to focusing more particularly on the teacher and his learning.

ANALYSIS AND DISCUSSION

As stated by Michael, it was his participation in the research project, a project involving tasks and technologies that were new to him, that led to new awarenesses regarding his practice of teaching algebra. From an analysis of the two observed lessons in conjunction with the follow-up interview, we too became aware of the changes that had occurred in Michael. This section focuses first on what changed in Michael and, second, on what enabled those changes.

What Changed in Michael

Zaslavsky and Leikin (2004) have pointed out that, by observing their students' work and by reflecting on this work, teachers learn through their teaching. We have found this to be the case for Michael as well. In particular, Michael's learning was in three areas: his knowledge of mathematics, his knowledge of mathematics teaching and learning, and his practice of teaching.

His knowledge of mathematics. Before his participation in the project, Michael had never really had the opportunity to think about a general rule for factoring the family of polynomials of the form $x^n - 1$. The prior workshop sessions between the members of the research team and the participating teachers had included discussions on this task and on one of the ways that its proving component might be thought about. However, the mathematics that Michael learned from engaging in the two class lessons on this task went beyond that which he had learned during the workshop. It involved specifically certain ways to approach the proof problem; moreover, this new learning on Michael's part was provoked by the students themselves.

The proof produced by Janet and Alexandra, which involved factoring by grouping with the generic example for $n = 8$, was one that had not occurred to Michael before his students actually generated it. He found it a 'nice little proof', to use his own words. A second contribution to the mathematical learning of Michael was occasioned by the proving attempt of Andrew. While Andrew was describing what he had observed for the CAS factoring of $x^{10} – 1$, it became clear that not only was this an interesting finding for his classmates, but also for the teacher, Michael. The new pattern that Andrew had noticed regarding the factoring of x^5+1 as $(x+1)(x^4-x^3+x^2-x+1)$ paralleled the pattern that he had noticed earlier regarding x^3+1 as $(x+1)(x^2-x+1)$. Although time did not allow for the proving of his conjectured new rule nor for its integration into the previous proofs that had been put forward during that lesson, there was no doubt that this was a new piece of mathematics for Michael.

His knowledge of mathematics teaching and learning. As reported by Zaslavsky and Leikin (2004), teachers' engaging in learning activities designed for student mathematical learning can be an effective vehicle for their professional growth. An additional factor that has been emphasized by Mason (1998) is that it is one's developing awareness in actual teaching practice that constitutes change in one's 'knowledge' of mathematics teaching and learning. Although Michael did participate in our professional development workshop prior to his integrating the novel tasks and technology into his teaching, it was his actual practice with these materials that had the greater impact regarding his 'developing awarenesses' in the area of mathematics teaching and learning. We focus on five of these new awarenesses.

* Michael developed a new awareness of what students at this grade level can accomplish mathematically – given appropriate tasks – as well as the realization that they can go further mathematically than expected (Extracts 4 and 5).

* Michael developed a new awareness of the role that technology can play in the mathematical learning of students (Extracts 5, 6, and 7).

* Michael developed a new awareness that students' mathematical knowledge changes with the combined duo of 'task-technology' (Extract 3).

* Michael developed a new awareness of how he might further provoke mathematical reflection in students; that is, that he could go even further than he usually went in his questioning, given appropriate tasks (Extract 4).

* Michael developed a new awareness regarding the culture of the class: it changes when technology is present and is used in the classroom. Students become more involved; they are more autonomous (Extract 6).

His practice in succeeding mathematics classes. It was not only Michael's awarenesses, which developed during this project. These new awarenesses were translated into practice. As he had said during his interview, he fully intended to take advantage of the wider scope offered by the project tasks and technology and to use them in subsequent years to push his students into thinking more deeply about their mathematics (Extracts 2 and 4). We continued to observe several of Michael's classes during the two years following this study. We witnessed, just as he had hinted he would do, a continuing development of his approach to encouraging students to go a little further in their thinking. This reflected his realization, which he stated during the interview, that he could push his students to think a little bit more about their mathematics. In addition, we also observed that he never stopped using the tasks and CAS technology that he mentioned he had so much enjoyed using during the present study. Thus, we were able to note a further evolution in his teaching practice – a practice characterized by the newly-acquired awareness of the role that technology, when accompanied by appropriate tasks, can play in the development of students' mathematical learning. Regular conversations with him during the ensuing years, including one quite recently, have highlighted his and his students' successes with, in particular, the $x^n - 1$ task with its proving component.

What Enabled These Changes

We can point to several factors that enabled the changes that we observed, as well as those that were disclosed to us by Michael himself. These enabling factors were found to include the following: a) Access to the resources and support offered by the research group; b) Use of CAS-supported tasks whose mathematical content differed from that usually touched upon in class; c) Michael's disposition toward student reflection and student learning of mathematics; d) The quality of the reflections of his own students on these tasks; e) Michael's attitude with respect to his own learning. The first two factors relate principally to the role played by resources 'from without', while the remaining three could be said to be 'from within' in that they concern the given teacher and his students. However, as will be argued, it is the interaction of the two dimensions that promoted teacher change.

Access to the resources and support offered by the research group. As was noted above, the change in Michael's knowledge of mathematics was occasioned by two different, but related, experiences. The first of these involved his prior discussions with members of the research group. These discussions had focused on new tasks and thus new mathematical awarenesses, which thereby constituted a first round of change with respect to Michael's existing mathematical knowledge. While the ideas for, and initial design of, the tasks came from the research group, these were shared with the participating teachers in a workshop setting that involved their working on the tasks themselves. They were then invited to offer feedback and to suggest changes to the tasks. These changes were subsequently integrated into a modified design for the tasks. Thus, the first exposure to the mathematical ideas inherent in the tasks occurred during the workshops that preceded the integration of the tasks into the teachers' classroom lessons. It was at these workshop sessions that Michael initially encountered the mathematics of the $x^n - 1$ task. This was also his first introduction to the use of CAS technology.

As the project unfolded and the researchers became a regular presence in Michael's class, there was ample opportunity to provide ongoing support to Michael. The researchers were able to offer assistance of a pedagogical, mathematical, and technical nature, whenever Michael so desired. In actual fact, such requests for support were quite rare, as each of the tasks was accompanied by a teacher version that included suggestions regarding discussion ideas, as well as additional information of both a mathematical and didactical sort. The normal interaction between Michael and the researchers after each lesson tended to be informal and conversational, much like that between collaborators.

Use of CAS-supported tasks whose mathematical content differed from that usually touched upon in class. Michael had expressed the fact that the $x^n - 1$ task with its exploration of the factors of this family of polynomials for integer values of *n*, along with its proving component, was a type of task that went far beyond what the students 'needed to know.' (It was also a new type of task for him.) While he might never have presented such a task to his students in the past, the experience of doing so convinced him that such tasks are indeed not only feasible, but also enjoyable to the students and lead to deeper mathematical reflection on their part (Extract 5). Based on our observations, we contend that novel tasks such as this one can change in a positive manner the usual teaching-learning dialectic of the mathematics classroom and are at the heart of both student and teacher learning.

Watson and Mason (2007) have argued that, "factors which influence the effectiveness of a task in promoting the intended kind of activity include … established practices and ways of working; students' expectations of themselves and of each other as influenced by the system and their pasts …" (p. 207). While 'novel tasks' may in fact be part of some teachers' "established practices and ways of working," we think rather that novel tasks – especially those involving proofs – in which algebra students have never before engaged, are likely the exception and not usually the norm. The very absence of "established practices" or student "expectations" may, in fact, lead to the success of novel tasks, and thus to some nuancing of Watson and Mason's statement.

Michael's disposition toward student reflection and student learning of mathematics. Michael worked very hard at encouraging his students to reflect, at giving them time to reflect, at listening closely to their reflections, and at having them share their reflections with the rest of the class. Even if he expressed the realization that, with the help of the activities designed by the research group, he could do even more in this regard, he was already predisposed to such practice. This predisposition was of course related to the importance Michael ascribed to students' learning to think for themselves. One of the signs of this didactical stance on mathematical learning was the way in which he often presented counterexamples to challenge students' thinking rather than immediately correcting them or giving the right answer. He aimed at having students develop their mathematical reasoning and critical thinking.

In her study of one teacher's practice of listening and responding to students' solution strategies, Doerr (2006) found that, "as the teacher asked for students to describe and explain their thinking, this not only contributed to the teacher's understanding of their thinking, but it created a situation where the students could refine their thinking and shift to a new way of thinking about the problem" (p. 20). As the case of Michael suggests, not only does listening to students support the development of students' thinking, it also leads to new awarenesses and professional growth in the teacher. Had Michael been a teacher who did not encourage the voicing of his students' mathematical ideas, he would hardly have come to know that, "their learning through the technology was huge" (Extract 7), nor would he have realized the pedagogical role that technological tools can play in enhancing mathematical learning. Thus, as Leikin and Levav-Waynberg (2007) have pointed out, a teacher's pedagogical principles can provide support for the growth of his/her knowledge regarding not only student learning but also teacher learning. In fact, Michael's pedagogical disposition with regard to mathematical activity (i.e., his view on encouraging the voicing of student reflection) served to beget not only new pedagogical knowledge regarding mathematics teaching and learning but also new mathematical knowledge for him.

The quality of the reflections of his own students on these tasks. During the interview, Michael mentioned on several occasions how struck he was by the quality of the mathematical contributions of his students, contributions such as those by Janet and Andrew, which had evoked new mathematical insights within Michael, as well as within the students of his class. He was clearly a teacher who could learn from his students, just as did the teachers in the Leikin and Levav-Waynberg (2007) study; these researchers reported that, "teachers who are sensitive to their students and flexible in their interactions with them, and who grant students autonomy in learning, end up learning mathematics from their students' replies" (p. 366).

In addition to Michael's development of mathematical knowledge from his interactions with his students, so too was the development of his knowledge of mathematics teaching and learning enhanced by the quality of his students' reflections. New awarenesses, such as, 'that which students at this grade level can accomplish mathematically' and 'the role that technology can play in the mathematical learning of students,' were occasioned by the students themselves. This is in contrast to Monaghan's (2004) findings that some teachers in his study noticed that, "tasks in technology-based lessons led their students to focus on the technology and at least three of the teachers felt an 'is this maths?' tension when their students attended to technological details at, in their opinion, the expense of the mathematics"(p. 336).

Michael's attitude with respect to his own learning. Each time that Michael said during the interview, "it's made me realize … ", we interpreted this to indicate Michael's openness to learning from his project participation. He experienced a great deal of joy – mentioned many times throughout the interview – at seeing how the students were positively responding to the tasks and, thus, in his learning not only about the mathematical levels they were reaching from their experience but also about the ways in which the technology and the tasks themselves were encouraging this response. He was in fact reflecting on his students' reflections. It is also noted that Michael was open to participating in the project right from the start, to learning something from it – even if he was not sure initially whether it would lead to new learning for his students. Watson and Mason (2007) have emphasized that, "to become an effective and professional mathematics teacher requires development of sensitivities to learners through becoming aware of one's own awarenesses" (p. 208). There is little doubt that the professional awarenesses Michael developed throughout the project, and which he shared with us, constituted a heightened sensitivity regarding his learners.

Reflections on What Changed in Michael and on What Enabled these Changes

We stated above that the changes in Michael's knowledge of mathematics, of mathematics teaching and learning, and in his practice of teaching were enabled by two kinds of factors, those from without and those from within. However, as our study progressed, it became clear that it was in the interaction of these two types of factors that change was promoted. Had it not been for the 'from-without' factors, that is, the access to the resources and support offered by the research group and, consequently, the use of CAS-based tasks whose mathematical content differed from that usually touched upon in class, then the 'from-within' factors, such as, the quality of the reflections of his own students on these tasks, would not have been put into play. Similarly, had it not been for 'from-within' factors, such as Michael's disposition toward student reflection and student learning of mathematics, as well as his attitude with respect to his own learning, then the 'from-without' factors related to the research team's contributions would not have taken root and flowered. Both types of factors supported each other in a mutually intertwining manner.

This is of interest from a theoretical perspective. It suggests firstly that the integration of novel materials and resources that have been designed to spur mathematical learning is more likely to be successful when the teachers who are doing the integrating are able to see that these resources are having a positive effect on their students' learning. Secondly, the novel materials and resources have a greater likelihood of producing this positive effect on student learning when the teacher doing the integrating engages in teaching practices that encourage student reflection and mathematical reasoning. The synergy between the two types of factors was found to be a positive force in the development of Michael's professional awareness, and one that constituted change not only in his knowledge of mathematics and mathematics teaching/learning, but also in his practice.

One final remark in this section concerns the role of the CAS technology on Michael's learning. As mentioned earlier, Michael's prior experience with classroom technology had included mainly graphing calculators, but not the CAS. Before the unfolding of the project in his own classroom, he never imagined the impact of this technology on his students' mathematical learning, and thus on his own learning of what his students could accomplish. At the heart of his coming to see that the student learning through the technology was huge was his realization that the presence of the technology changes the nature of the questions that can be asked of students, and thus the kind of mathematical reflection they engage in. While the tasks themselves were a crucial component of Michael's learning within his own practice, the actual design of the tasks was set up in such a way as to work hand-in-hand with the affordances of the technology. In fact, the first two parts of the $x^n - 1$ task set, which were foundational to the proving part of the activity, could not have been managed without the CAS. In this respect, the CAS technology was central to Michael's learning.

CONCLUDING REMARKS

In conclusion, we wish to emphasize briefly only two issues. One is that, while this study fits into the broad research domain of teachers learning from their own practice (e.g., Jaworski, 2006; Zaslavsky & Leikin, 2004), a significant feature has been that that practice was nourished by input coming to a large extent from outside. The second issue concerns the mathematical activity that was stake in the study, that of proving.

With respect to the first issue, much of the research related to teachers' learning from their own practice emphasizes teachers' planning of their interactions with students, followed by their subsequent reflective analysis of these interactions. Considerably fewer studies (exceptions include, e.g., Leikin, 2006) follow the path that we did where the majority of the planning of the instructional interaction with respect to the mathematical content and the task questions to be posed to the students had already been elaborated in advance by the research team, even if in partial collaboration with the participating teachers. This, we feel, added a dimension to the study that does not often come into play in research on teaching practice. As a consequence, the teacher's reflective analysis of his interactions with the students had to take into consideration – in a somewhat different manner than would otherwise be the case – the worthiness, or not, of the particular mathematical content at stake, the way in which it was elaborated, and the technological tools that were used to support its approach. The positive nature of the reflections shared by Michael during the post-lesson interview with one of the researchers suggests that the integration of resources coming from without can be a powerful stimulus to teachers' learning from their own practice.

With respect to the second issue, only rarely does the teaching of algebra in high school include activity with proving. The teacher featured in this study, Michael, could be said to have been very courageous in agreeing to integrate into his teaching of algebra the $x^n - 1$ task with its proving component. He had never before included proofs within his algebra teaching; nor had his students ever engaged in this form of algebraic activity. Nevertheless, the success that he and his students experienced with it went way beyond his (and likely their) expectations. Hanna and Barbeau (2008) have raised the following query: "Approaching proof as more than a formal way of certifying a result is bound to make increased demands on the teacher and involve more engagement by the students; the long-term value would seem to be clear, though not quantified, but can the increased demands be managed?" (p. 352). Michael's and his class's experience with the proving segment of the $x^n - 1$ task provides a strong existence proof of the notion that the increased demands can indeed be managed.

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