BRYAN'S STORY: CLASSROOM MISCOMMUNICATION ABOUT GENERAL SYMBOLIC NOTATION AND THE EMERGENCE OF A CONJECTURE DURING A CAS-BASED ALGEBRA ACTIVITY[1]

Ana Isabel Sacristán   &   Carolyn Kieran

Center for Advanced Studies & Research (Cinvestav), Mexico   Université du Québec à Montréal, Montréal, Canada

In this paper we describe a high-school student's difficulties in understanding the notation for a general polynomial: $x^{n-1} + x^{n-2} + ... + x + 1$ during a CAS-based activity to develop a general factorization for $x^n - 1$. We illustrate how he attributes a meaning to the ellipsis symbol related to his experience of "terms cancelling out" rather than the taken-for-granted meaning of an undefined continuing process. We show his associated difficulties in making sense of the sequence of decreasing exponents. Finally, emerging from his misunderstandings, we describe his motivation to find a general formula for $x^n + 1$, testing his conjectures using the CAS.

INTRODUCTION

In this paper we describe the story of a 15-year-old high-school student, Bryan, working on a CAS-based algebra activity: we show some of the issues that arose in terms of language and communication difficulties, and how he used the tool in his quest for meaning and to generate and test conjectures. We are interested in narrating Bryan's story and the evolution of his thinking process, for two reasons. First, it highlights possible language and communication problems in a classroom: the difficulties that students may face when encountering, for the first time, conventional symbols (that can be taken for granted by teachers) and general algebraic notations, and the "cross-talk" that can happen between teachers and students. There are conventions which we take for granted and we do not realize might be a problem for students. In this particular case, general expressions of polynomials using the ellipsis ('…') notation were a source of difficulties. Language and communication in the mathematics classroom and the use of symbols have been extensively studied (e.g., Pimm, 1987, who does address the issue of symbols from common writing systems that are used in mathematics with perhaps different conventional meaning), but nowhere in the current research literature could we find a discussion or case related to the ellipsis notation. Second, the story narrated here, is one that took place in a CAS-based activity using TI-92 Plus calculators. What is thus also interesting is how, in face of his confusions, the student was motivated to make use of the tool and test a conjecture that arose from his misunderstanding.

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THE STUDY

This report emanates from an ongoing research study whose central objective is the shedding of further light on the co-emergence of technique and theory (e.g., Artigue, 2002) within the CAS-based symbol manipulation activity of secondary school algebra students. Several sets of activities that aimed at supporting this co-emergence were developed by members of the research team[2]. Six 10th grade classes were involved in the study. All of their classroom-based CAS activity was videotaped; and digital records were made of all the student worksheets. The results presented here focus on a selected aspect of the analysis from one of the activity sets (see Kieran & Saldanha, 2006; see, as well, a companion research report relating to the same study, but presenting a different component of the analysis).

The Factoring Activity

The factoring activity (inspired by an example developed by Mounier & Aldon, 1996, and described by Lagrange, 2000) had as objectives to establish connections between notions that students already knew regarding the difference of squares and the sum and difference of cubes:

$$x^2 - 1 = (x-1)(x+1); \quad x^3 - 1 = (x-1)(x^2 + x + 1) \quad ; \quad x^3 + 1 = (x+1)(x^2 - x + 1),$$

and the general factorization for $x^n - 1$ given by:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1).$$

We designed a worksheet with a sequence of tasks for this activity, alternating with reflection questions and whole-class discussion periods. In many of those tasks, students were asked to factorize particular cases of the type $x^n - 1$ first using paper-and-pencil, then the calculator, and then to show how the paper-and-pencil and the CAS results could be reconciled.

BRYAN'S STORY

During the factoring activity, the students had been working on factoring the cases: $x^2 - 1; \ x^3 - 1; \ x^4 - 1$. It was intended that they begin to see a general pattern that they might not have noticed when initially learning to factor a difference of squares and a difference of cubes. This was followed by a question (see Figure 1), a class discussion, and predicting the factorization of $x^5 - 1$. During the class discussion, the teacher showed on the whiteboard that when factoring each of the above expressions, the product of the factors would make all the middle terms "cancel out": for example, for $x^2 - 1$: $(x-1)(x+1) = x^2 + x - x - 1$. Also in this discussion, when the teacher asked

who could say something about all those expressions, it was Bryan who remarked that $x-1$ was a factor in all of them. Bryan was a student who participated and contributed very often in the class activities (at least in the CAS-based activities we observed) and always asked the teacher or classmates when he did not understand something. But he was also a bit of an attention-seeker; this caused him to not always be taken seriously by his classmates and the teacher, despite sincere doubts and smart conjectures.

2. (d) How do you explain the fact that the following products result in a binomial: two binomials, a binomial with a trinomial, and a binomial with a quadrinomial?

Because the middle terms always cancel out in this case which creates binomials

Figure 1: Bryan's answer to question 2d – "because the middle terms always cancel out in this case which creates binomials".

Towards a Generalization and the Introduction of New Symbols: The Problem of the Ellipsis

The question that followed was: "Explain why the product $(x-1)(x^{15} + x^{14} + x^{13} + ... + x^2 + x + 1)$ gives the result $x^{16} - 1$?" Here, Bryan's difficulties began. He exclaimed: "*The next [question] does not make any sense.*" As we will see, Bryan did not understand the meaning of the ellipsis ("…"). The teacher, however, did not pay attention to what his confusion could be; he simply explained what the task question asked: "*The middle terms cancel out, like we said before, like the thing we did before. So the 'minus one' takes care of all the terms apart from the first one and the last one. So, we're left with $x^{16} - 1$*". So Bryan wrote that as his answer (see Figure 2). A similar question was given for $x^{135} - 1$, and the reply was the same. As we will see, this repeated emphasis on the idea that "the middle terms cancel out" would have repercussions on the meanings created.

2. (f) Explain why the product $(x-1)(x^{15} + x^{14} + x^{13} + ... + x^2 + x + 1)$ gives the result $x^{16} - 1$?

The middle terms all cancel out

Figure 2: Bryan's answer to question 2f – "The middle terms all cancel out".

During the class discussion period that followed, the teacher introduced the general expression for $x^n - 1$ (earlier than was expected in the design of the activity). With the input of some students, he wrote on the whiteboard: $(x^n - 1) = (x - 1)(x^{n-1} + x^{n-2}... + x + 1)$. However, several students were confused by the notation and asked what n meant; the teacher explained: "*n means any integer, any positive integer.*" While writing the expression, the teacher read for the ellipsis "*plus 'dot dot dot'*". Bryan expressed his confusion on the meaning of this symbol:

Bryan: What are the dots?

Teacher (not really answering Bryan): So it goes all the way to one [pointing to the 1 in the expression: $x^{n-1} + x^{n-2}...+ x + 1$] Is that clear? We all see that?

Bryan: No, I do not understand. Why the dot, dot, dot?

Teacher (not really answering Bryan): It is the same way we did for x to the 135 [for $x^{135} - 1$]: you know that the middle terms are going to cancel out. […]

Bryan: What are the dots? Wouldn't it all be the same if it's x^{n-1} plus 1 [$x^{n-1}+1$] except we have that middle term thing; wouldn't that be the same thing?... If the first bracket is like $(x-1)$, but the second bracket, instead of putting that middle term thing in there, you just do x^{n-1} plus 1 [$x^{n-1}+1$].

[The teacher wrote out what Bryan said: $(x-1)(x^{n-1}+1)$, then showed that the two expressions were not the same; but Bryan was not satisfied.]

Teacher: So those don't cancel, do they?

Bryan: They should, though!

All the emphasis in previous tasks on cases where the middle terms cancelled out, seems to have led Bryan to give this meaning – that the middle part cancels out -- to the ellipsis symbol ('…'), which he did not understand. Thus the teacher and Bryan gave different meanings to the term 'cancel' and to the ellipsis, which, as we will show in the next sections, caused problems in the understanding of the algebraic notation. As Arzarello (1998, p. 259) points out:

One of the main problems in teaching algebra (and most of mathematics) is a communication problem. The relationship between signs and their mathematical meanings may be confused for many students who attach only formal and procedural features to the former but who use the same words as their teachers, albeit with different meanings, for representing the situation.

The teacher took for granted that the ellipsis symbol would be understood. In the history of mathematics, Cajori (1928/9, vol. II, pp. 59-60) reveals the following with respect to the use of general notation and the ellipsis symbol:

L'Abbé de Gua [1741] writes a finite expression, marking the terms omitted, with [four] dots and also with "&c.":   "3,4,5....&c *n-m*+2"
[with a bar over the *n-m*+2], the commas indicating here multiplication. F. Nicole [1743] writes a procession of factors, using dots, but omitting the "&c". [text deleted] C.F. Hindenburg [1779] uses dots between, say, the fourth term and the *n*th term, the + or the – sign being prefixed to the last or *n*th term of the polynomial. [text deleted] E.G. Fischer [1794] writes a finite expression $y = ax + bx^2 + cx^3 + ...+ px^r$ and, in the case of an infinite series of positive terms, he ends with "+etc.".

Cajori also adds that, "Descartes [1637] wrote a^3, x^4; the extension of this to general exponents a^n was easy" (Cajori, 1928/9, Vol. 1, p. 360). However, it is not clear when general polynomial notation and the use of the ellipsis symbol became widely accepted or even standardized, for Euler (1797, Vol. II, p. 31) in his *Elements of Algebra* was still using notation, for general expressions, such as

"$a + bx + cy + dxx + exy + fx^3 + gxxy + hx^4 + kx^3y + \&\mathrm{c}. = 0$" in compound indeterminate equations at the end of the 1700s.

However, when analysing our data, we realized that the ellipsis symbol is hardly ever defined. In dictionaries, the ellipsis is always described as something that is omitted or left-out; as a mathematical notation, in the online Wikipedia encyclopaedia (http://en.wikipedia.org/wiki/Ellipsis, retrieved 30 November, 2005) it does say that in mathematics the ellipsis is used to mean "so forth" to follow a pattern, but it is almost never defined in mathematics textbooks, even though it is used extensively, particularly for sequences and infinite processes. Even in books like Lakoff and Nuñez's (2000) that extensively discuss infinite processes, and that also focus on the meaning and understanding of mathematics and mathematical symbols, we could not find a discussion of the ellipsis symbol, except referred to as "the common mathematical notation for infinity" (p. 180).

So Bryan, when faced with this new symbol had to rely on the experience of the previous tasks, thus relating the ellipsis to the 'disappearance' of terms, something that is 'cancelled out', rather than a continuing process of existing terms that have to be omitted due to the generalization. As Pirie (1998) explains, the growth of mathematical understanding occurs through a process of folding back to earlier images to give insight to the building of new, more powerful ideas and that mathematical symbolism is open to interpretation only through the medium of verbal language, which relates the mathematics to the learner's previously comprehended metaphors, where the rift between meaning and understanding can occur. This misinterpretation of the ellipsis symbol by Bryan would be a problem that would continue throughout the activity as shown below.

On the other hand, it is interesting that this misinterpretation of the ellipsis symbol led Bryan to focus on the expression $x^k + 1$. In parallel with trying to gain clarification on the meaning of the general notation $\left(x^{n-1} + x^{n-2} \ldots + x + 1\right)$, he began to explore expressions of the form $x^k + 1$ on his CAS, as we will illustrate in the next sections.

Further Symbology Difficulties: Making Sense of the Continuing Process Described by the Ellipsis and by the Sequence of Exponents in the General Expression

Following Bryan's exclamation that the expression $(x-1)(x^{n-1}+1)$ should cancel out, the teacher tried to explain what made the terms cancel out in the general expression $(x-1)(x^{n-1} + x^{n-2} \ldots + x + 1)$, but this just added to Bryan's difficulties:

Teacher:	For it to cancel, these need to go numerically with the powers decreasing each time. So that's why you get 1…
Bryan:	So if it's decreasing, how far do you go? 'Til…?
Teacher:	You go all the way down. […]
Bryan:	What if …, if it's x^{n-1}, then you do x^{n-2}, then you do x^{n-3}, how far down will you go?
Teacher:	You go all the way 'til you get to x to the zero, which is 1.

Bryan:	Which is x to the n minus...?
Teacher:	[n] minus n
Bryan:	x to the n minus … but what do you get there, how do you know that?

Bryan was now very confused by the decreasing exponents. He could not see how you could get to 1, to x^{n-n} — a difficulty which is of course related to the difficulty understanding the ellipsis: since for Bryan the meaning of the ellipsis was "cancels out", and not a continuing process where something is simply omitted, he was unable to see the sequence of exponents. The teacher tried to explain with a numerical example; then went on to the next task. But Bryan remained very confused shaking his head; at one point he grabbed the calculator apparently to use it in his search for meaning and to test his claim that an expression containing x^k+1 as a factor should indeed 'cancel out.'

While individual work continued on the tasks, Bryan called the teacher over. He felt he had found a case, $x^{16}-1=(x^8-1)(x^8+1)$, that could 'prove' his supposition:

Bryan:	Sir? If its $x^{16}-1$, wouldn't that be the same as x^8-1 brackets x^8+1, close brackets. So, what I said was, but it's only if the term is….
Teacher:	Ok, what you've said is going to be very useful to what we're doing … You're one step ahead as usual, eh? … [Then to the whole class]: Those of you who heard what Bryan just said, it is very relevant for a change.

The teacher did not realize Bryan's confusion. He only picked up the correct ideas in Bryan's reasoning because he thought Bryan was applying the difference of squares method. But it seems that Bryan was trying to find a case not only where a term of the factored expression was of the form x^k+1, he wanted to go further than this. This became evident when the teacher later asked Bryan to show his method to the class for the case x^4-1. Bryan explained it as: "*the power number, you could divide it or something*". While classmates suggested that this was a difference of squares, Bryan clearly was still engrossed with the x^k+1 form, because he kept suggesting to further factor the term "*with the plus in the middle*."

Later, a group discussion followed trying to answer (from another question on the worksheet), for what values of n the complete factorisation of x^n-1 would: (i) contain exactly two factors; (ii) contain more than two factors; (iii) include $(x+1)$ as a factor. When the teacher said: "*So, you should have gone beyond the initial conjecture that it's just* $(x-1)(x^{n-1}+x^{n-2}...)$*,*" Bryan exclaimed: "*Sir, I don't understand that*". Bryan was still obsessed with trying to understand the notation. He stepped up to the whiteboard; and asked

Bryan:	"When you go: x minus 1 and then you continue on, right?, for a long time, trying to cancel them out, x minus 2, then you go x to the minus 3 [but he wrote '$n^{-1}+n^{-2}+n^{-3}$'], how do you know when this [circling the last exponent] is zero? How do you know when it is zero?"

Despite his language mistakes, he seemed to be asking when the sequence of exponents $n-1$, $n-2$, $n-3$ in the expression $x^{n-1}+x^{n-2}+x^{n-3}...$ would become zero.

Teacher: "It depends on n. That is just how you write it".

Bryan "But… When you go like this [writing an ellipsis] all the time?"

The teacher simply assented with a yes, misunderstanding Bryan's question; Bryan gave up asking, though he was still confused and kept shaking his head. Clearly he still did not understand the ellipsis notation or how the sequence of powers is defined. The next session, the teacher began by re-writing on the board the general formula $(x^n - 1) = (x - 1)(x^{n-1} + x^{n-2} + ... + x + 1)$; Bryan immediately expressed his concern:

Bryan: I don't like that.

Teacher: That's right [referring to the formula], isn't it Bryan? But you don't like that, because?

Bryan: Because I don't like the dots. I don't think it is a real answer"

Teacher: It's a general situation.

The Use of the Tool to Explore the Factorization of $x^k + 1$

The last task of the activity was to prove or explain why $x + 1$ is always a factor of $x^n - 1$ for even values of $n \geq 2$. For a while the students worked on their calculators, then they shared their 'proofs' in a group discussion. While a couple of students went to the board to explain their 'proofs,' Bryan was very restless. He desperately wanted to show what he had been working on.

When invited to the board himself, he picked up the difference of squares method that he had used in the previous session. He wrote that, if n is even, $x^n - 1 = (x^{n/2} - 1)(x^{n/2} + 1)$. He asserted that the first factor would be able to be factored further like they had done in previous tasks, but his focus was on the second factor. He explained that $x^n + 1$ for n being 4, for example, would equal $(x + 1)(x^3 - x^2 + x - 1)$ and would therefore have $x + 1$ as a factor. He said that he had tried it out and it worked. But some of his classmates and the teacher pointed out that that was not correct, and the teacher erased it. It was clear that Bryan had been trying to factor cases of the form $x^k + 1$ on the calculator and had noticed that when it factored out, he would get a factor with alternating signs. His problem was that he picked the wrong example (n=4).

Another student, who had been sitting beside Bryan during the previous class session, came forward to pick up the argument. He chose to illustrate the conjecture regarding alternating signs with the example $x^{10} - 1$, which when fully factored yielded $(x - 1)(x + 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1)$, thus showing indeed that $x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1)$, and that $x^k + 1$ is refactorable for certain values of k.

FINAL REMARKS

In this paper we have illustrated some difficulties that can arise in understanding the general notation of polynomials of undefined degree. In the case of Bryan, there were two difficulties: a misunderstanding of the meaning for the ellipsis symbol (not discussed elsewhere in the literature) and difficulties in making sense of the general

sequence of decreasing exponents. In the end, we do not know whether Bryan's problems with the ellipsis notation were solved, but we did notice a marked improvement in the use of the general notation $(x^{n/2}-1)(x^{n/2}+1)$, which we attribute to his explorations with the calculator. Berger (2004) has argued, that the meaning of a new mathematical sign that is presented to a student, evolves through communication and functional use of the sign in mathematical activities embedded in a social context; so we would expect that Bryan eventually will make sense of that symbol. On the other hand, it is very interesting how the support of the CAS calculator, in conjunction with the task questions and the group interactions, allowed Bryan and his classmate to work toward a new general expression that had not been foreseen: that, for n odd, $x^n+1=(x+1)(x^{n-1}-x^{n-2}+...-x+1)$. (Further discussion of the x^n+1 conjecture is found in Kieran and Drijvers (2006).) We do not believe the motivation and ability to generate and explore this conjecture would have been possible without the support of the CAS.

Acknowledgments

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