A SLIPPERY SLOPE BETWEEN EQUIVALENCE AND EQUALITY: EXPLORING STUDENTS' REASONING IN THE CONTEXT OF ALGEBRA INSTRUCTION INVOLVING A COMPUTER ALGEBRA SYSTEM

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This study explores the reasoning of high school students who participated in a classroom teaching experiment in which a computer algebra system was used to promote reflection on equivalence of algebraic expressions. Results of a post-test indicate that many students experienced difficulties teasing equivalence and equality apart. Moreover, results suggest that reasoning about expressions in terms of their having a common form played a significant positive role in enabling students to ascertain whether given expressions are equivalent.

Background and Perspectives

Although the idea of equivalence of algebraic expressions lies at the heart of transformational activity in algebra, it is not typically placed at the foreground of algebra instruction. The research literature on algebra learning, while extensive (Kieran, 1992), includes relatively few inquiries into students' understandings of algebraic equivalence (Ball, Pierce, & Stacey, 2003; Kieran, 1984; Pomerantsev & Korosteleva, 2003; Steinberger, Sleeman, & Ktorza, 1990). Consequently, relatively little is known about students' understandings of algebraic equivalence and, in particular, how those understandings might play out in the sense they make of transformational work in algebra.

In a comparison of novice and more advanced high school students' performance on tasks dealing with equivalence of linear equations, Kieran (1984) found that the latter group showed little awareness that the equation-solving processes they were able to execute proficiently preserve solutions. In a study using a much larger sample of eight-and ninth-grade students, and employing tasks similar to those used by Kieran (1984), Steinberg, Sleeman and Ktorza (1990) found that while most students knew how to use transformations to solve simple linear equations, many did not spontaneously relate this knowledge to the production of equivalent expressions. Ball, Pierce and Stacey (2003) developed an instrument designed to assess students' abilities to quickly recognize equivalent algebraic forms—the "Algebraic Expectation Quiz" (ibid., p. 4). On the basis of the performance of a sample of fifty students on this test, administered before and after students' progression from 11^{th} -to 12^{th} -grade and during which time they participated in algebra instruction involving the use of a computer algebra system (CAS), the researchers reported that "recognizing equivalence, even in simple cases, is a significant obstacle for students" (ibid., p. 4). Pomerantsev and Korosteleva (2003) presented compelling evidence that difficulties related to understanding algebraic equivalence can extend well beyond the post-secondary level. Their study (ibid.) involved the use of a diagnostic test administered to a large (N=416) sample of students enrolled in different stages of their K-8 teacher preparation program at a major American university. Test items were designed to assess students' abilities to discern and use structural aspects of algebraic expressions; results of this research revealed serious difficulties in doing so, cutting across students in all groups contained in the sample.

Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). (2005).
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Within the last decade or so a fairly coherent research program has emerged out of the French *didactique* tradition of research in mathematics education. A number of these studies have explored the use of computer algebra systems (CAS) in mathematics classes at the high school or college level (e.g., Artigue, 2002; Guin & Trouche, 1999; Lagrange, 2000). These researchers argue that CAS can be used as a tool to promote the co-development of both conceptual understanding and technical proficiency among students, provided that technical aspects of mathematical activity are not ignored. For instance, Lagrange (2000) frames the idea of mathematical technique as a bridge between task and theory, in the sense that as students develop techniques in response to certain task situations, they concomitantly engage in theory-building and reflection on the technical aspects of their activity in relation to the mathematical ideas addressed in the task. In a study touching on the idea of equivalence, Artigue (2002) drew on students' work involving the passage from one given form of expression to another to illustrate how the research team attended specifically to the fact that "equivalence problems arise which go far beyond what is usual for the classroom" (p. 265). Artigue's study employed a CAS as a "lever to promote work on the syntax of algebraic expressions, which is something very difficult to motivate in standard environments" (ibid.), asserting that it forces students to confront issues of equivalence and simplification. Similarly, Nicaud et al. (2004) foreground the importance of equivalence in algebra, framing it as "a major reasoning mode in algebra; it consists of searching for the solution of a problem by replacing the algebraic expressions of the problem by equivalent expressions" (pp. 171-2).

It is against this backdrop, which highlights the importance of the notion of algebraic equivalence as well as students' reported difficulties therein, that we situate the current study.

The Study

Goals and Participants

Our study is part of an ongoing three-year project involving 5 classes of 10^{th} -graders who have been following an integrated curriculum involving algebra as part of the course of study since Grade 7. Inspired by, and wishing to build upon, the recent research emerging out of France, a central aim of this project is to inform our understanding of high school students' emerging ideas about algebra in relation to their engagement with instruction that integrates the use of CAS with traditional paper-and-pencil work. More specifically, the aim of this study is to gain insight into students' reasoning about the idea of equivalence in the context of instruction designed to foster the emergence of such reasoning.

The particular group of 15 students featured in this brief report comprised a class at a private school in a major metropolitan center in eastern Canada. The students had learned basic techniques of factoring and solving linear and quadratic equations during the previous year and had used graphing calculators on a regular basis. However, they had no prior experience using symbol-manipulating calculators. Results of a pre-test indicated that these students were quite skilled in algebraic manipulation.

Method of Inquiry

Our research team conducted teaching experiments in these classes. This entailed engaging students in a sequence of eight instructional activities designed to fit into their program of study, which was taught by their regular classroom teacher, and to integrate TI-92 Plus calculators with traditional paper-and-pencil algebra work.

The idea of algebraic equivalence was an underlying conceptual thread running through the design of the activities, with a particular subset of three of the activities designed specifically to support student reflection on this idea explicitly in relation to the transformation of expressions and the solving of equations. The activities were each designed to take up to two class periods. Each activity was punctuated by parts, each part included presentation of student work and discussion of the main issues raised by the tasks in the given part. Tasks were of three types that involved either work with CAS, or with paper/pencil, or were of a reflective nature. Each activity was accompanied by a teacher version that included suggestions for classroom discussion. In designing these tasks, the research team took into serious consideration both the students' background knowledge and the fact that the tasks were to fit into an existing curriculum; but we also moved to ensure that they would unfold within a particular classroom culture that gave priority to discussion of serious mathematical issues.

The data corpus generated for this study includes: audio-video recordings of the classroom lessons centered around the activities and of individual interviews conducted with selected students after each lesson; students' written work on activity sheets and a post-test; field notes generated by members of the research team who were present during the unfolding of the classroom lessons.

Addressing Algebraic Equivalence: Foregrounding the Idea of Common Form

The sequence of 3 activities in which the idea of equivalence was explicitly addressed unfolded over 4 hour-long classroom lessons held on consecutive days, starting approximately five weeks into the academic year. The research team designed these activities in consultation with the classroom teachers who implemented them. These consultations together with the team's examination of the Grade 10 mathematics textbook used in the class indicated that the activities would constitute students' first encounter with equivalence and its relation to algebraic expressions, transformations, and equations as an *explicit* idea of reflection.

In broad terms, the intended conceptual progression of these activities was to have students develop connections between equivalence of expressions, addressed in the first two activities, and equivalent equations, addressed in a third activity. Details of this sequence entailed having students use numerical evaluation of expressions, and comparison of their resultant values, as the entry point for discussions of equivalence. The impossibility of testing all possible numerical replacements in order to determine equivalence motivated the use of algebraic manipulation and the explicit search for common forms of expressions—an idea highlighted in classroom discussions. Discussions also included attention to restrictions on equivalence. The relation between equivalent/non-equivalent expressions and equation solutions was then explored in both CAS and paper-and-pencil tasks. An outline of the content sequence of these activities is shown in Figure 1 (Kieran & Saldanha, 2005). In these activities, the idea of *equivalent expressions* was eventually defined formally as follows: "We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values".

Activity 1: Equivalence of Expressions
- Part I (with CAS): Comparing expressions by numerical evaluation
- Part II (with paper/pencil): Comparing expressions by algebraic manipulation
- Part III (with CAS): Testing for equivalence by re-expressing the form of an expression – using the EXPAND command
- Part IV (with CAS): Testing for equivalence without re-expressing the form of an expression – using a test of equality
- Part V (with CAS): Testing for equivalence – using either CAS method

Activity 2: Continuation of Equivalence of Expressions
- Part I: Exploring and interpreting the effects of the ENTER button, and the EXPAND and FACTOR commands
- Part II: Showing equivalence of expressions by using various CAS approaches

Activity 3: Transition from Expressions to Equations
- Part I (with CAS): Introduction to the use of the SOLVE command
- Part II (with CAS): Expressions revisited, and their subsequent integration into equations
- Part III (paper/pencil): Constructing equations and identities
- Part IV (with CAS): Synthesis of various equation types

Figure 1: Outline of content of the three activities addressing equivalence.

Shortly after completing this sequence of activities, students took a post-test designed to query their understandings of the content addressed within these activities. A post-test question pertinent to the coming discussion is shown in Figure 2.

Q.5 The following equation has $x = 2$ and $x = 2/3$ as solutions:

$$x(2x-4)+(-x+2)^2 = -3x^2+8x-4$$

(i) Precisely what does it mean to say that, "the values 2 and 2/3 are solutions of this equation"?

(ii) Use the CAS to show that: (a) the two values above are indeed solutions, and
(b) there are no other solutions.

What I entered into the CAS	What the CAS displays and my interpretation of it

(iii) Are the expressions on the left- and right-hand sides of this equation equivalent? Please explain.

Figure 2. A post-test question (see part iii) that revealed interpretation of equivalence.

Data Analysis and Results

Our report focuses on results of a preliminary analysis of students' post-test responses.[1]

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[1] See Kieran & Saldanha (2005) for a discussion of students' interaction with the CAS in relation to their emerging understandings of equivalence. In that report we describe the unfolding of the activity sequence and selected instructional episodes in greater detail.

Analytical Method

Analysis of these data unfolded in a manner that initially involved mutually influential and overlapping phases, but that eventually coalesced into more independent and identifiable broad stages. Following Saldanha (2004) and Thompson, Saldanha & Liu (2004) we began, before examining any data, by identifying relevant dimensions of each question to which students who had engaged in the activity sequence might be expected to attend. Two overarching criteria were used to determine whether a dimension of a response to a particular question was deemed relevant, for our purposes: the dimension ostensibly reflected 1) a certain sensitivity to a key idea addressed in instruction, and 2) aspects of the kind of understanding targeted in instruction—that is, understandings of the idea(s), addressed in particular question, that we considered to be desirable and in line with the aims of instruction. Our initial determination of relevant dimensions was then revised and refined through a process of negotiation as the research team began to examine and take into account some of the post-test data. The set of relevant dimensions for each post-test question converged to a final form as our examination of the student data became increasingly directed and systematic.

In a final stage of analysis each of the authors and a third member of the research team independently coded all students' responses to the post-test questions, for correctness and for whether they indicated correct or incorrect attention to the relevant dimensions. For instance, for part (iii) of Question 5 of the post-test (see Figure 2) we considered three relevant dimensions of a response: a) its correctness; b) appeals (implicitly or otherwise) to the idea of numerical equality of both expressions for each numerical replacement value of x; c) appeals to the algebraic idea of common form.

Initial disagreements in independent code assignments were few and usually rooted in differing interpretations of student explanations that lacked *explicit* reference to a key idea. These disagreements were resolved by a process of comparison and negotiation and a 100% consensus on the coding was achieved.

We should add, further, that while this scheme constituted the bulk of our method for a first analysis of the data, it was also accompanied by a less systematic documentation of student responses. Here we flagged selected responses as indicative of conceptions that appeared to depart significantly from those we envisioned as instructional endpoints.

Result 1: Disentangling Equivalence and Equality

Our analyses of post-test responses revealed that many students' emerging thinking about equivalence was tightly bound up—indeed, arguably confounded—with notions of numerical equality. This is illustrated by the following, incorrect, response to Question 5(iii) of the post-test, shown in Figure 3:

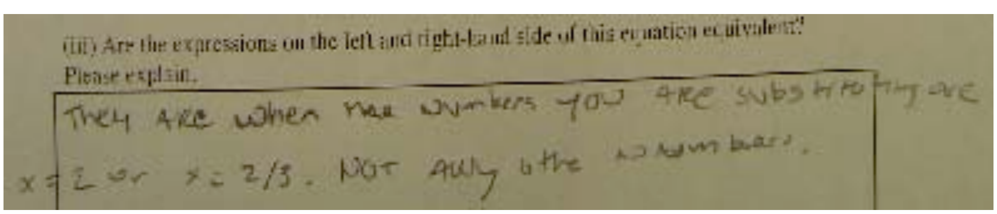

Figure 3. A response suggesting a murkiness between equality and equivalence.

We detect in this illustrative response a certain precariousness between the ideas of equality and equivalence. The response is consistent with someone having assimilated the idea of equivalence into a conception rooted exclusively in numerical equality: this becomes clear if the reader simply replaces the word "equivalent" with "equal". On the negative side, the response contains little evidence of an understanding of equivalence as a structural property of pairs of algebraic expressions. There is also no evidence of an understanding of algebraic transformations as an indispensable tool for converting either/both expressions into identical form and thereby eliminating the need to demonstrate equality for all values of x by exhaustive evaluation and comparison.

The response above is in apparent contrast to those like the two shown in Figure 4, both of which suggest attention to parts of these ideas and a clearer distinction between equivalence and equality:

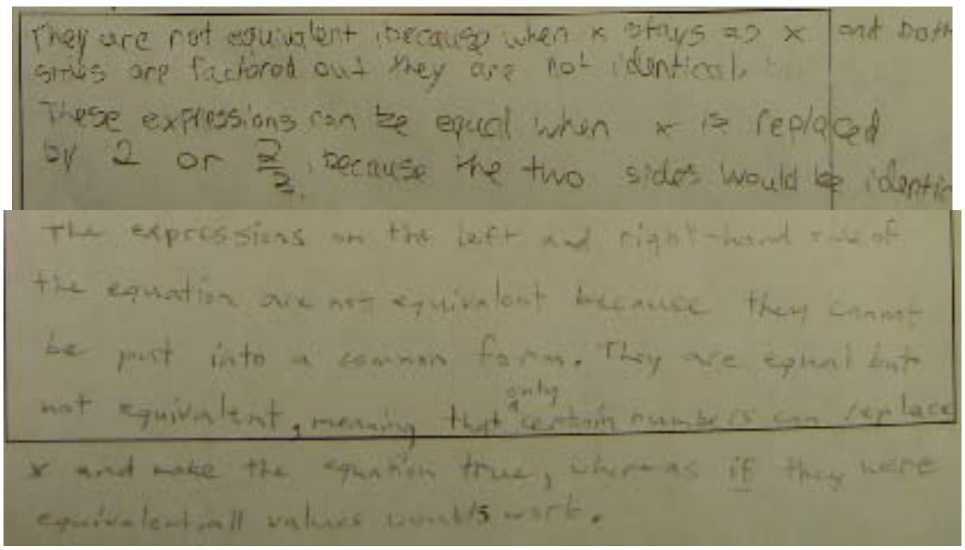

Figure 4. Responses suggesting clearer understandings of the relationship between equality and equivalence.

Result 2: The Role of Common Form

In addition to these results, we examined the distribution of students' responses to Question 5(iii) in relation to the idea of common form. Our findings, tabulated in Figure 5, reveal a compelling correlation: a very large proportion (89%) of students who gave a correct response made some correct reference to the idea of common form in the post-test, while none (0%) of those who answered incorrectly did so.

	Correct response	Incorrect response	Total
Correct reference to common form	8	0	8
Incorrect reference to common form	0	2	2
No reference to common form	1	4	5
Total	9	6	15

Figure 5. Distribution of students' responses to post-test Question 5(iii) (see Figure 2) in relation to references to common form.

The results presented in Figure 5 indicate that the idea of expressions having a common form, when understood well and used correctly, played a significant positive role in students' abilities to assess equivalence of expressions. Indeed, it would appear that sound reasoning about common form was an almost sufficient condition for enabling students to correctly assess equivalence of expressions.

Result 2 is encouraging in its suggestion that our instructional sequence oriented a significant proportion of students to common form as a salient idea and to its role in understanding algebraic equivalence. At the same time, however, Result 1 suggests that this was not the case for a somewhat smaller, though arguably still significant, proportion of students. Two questions naturally arise for us as a result of these mixed findings; the first is how the thinking of students in these two groups might differ significantly in ways that could account for the differences we documented. A second, perhaps related, question is whether aspects of the instructional sequence and engagement with it might have hindered some students in making the conceptual advance from equality to equivalence that we intended.

These two clusters of results now point us in a clearer direction for a next phase of our study: they suggest that we need to consider in greater detail, and by triangulation with our other data sources, how the thinking of students in these two groups might differ significantly. In addition they impel us to question our instructional design assumptions and decisions with regard to both the specific tasks we set to support students' transition from equality to equivalence and the engagements we envisioned around those tasks.

Conclusion

Findings from our preliminary analysis of students' written post-test responses suggest, on one hand, that the idea of common form—a notion that was given primacy in instruction—played a significant positive role in helping students assess the equivalence of given algebraic expressions. On the other hand, we also found ample evidence that distinctions between equality and equivalence were, at best, murky for many students. This murkiness was present despite their having participated in an instructional sequence designed specifically to support their clarifying the distinctions and negotiating the conceptual advance from equality to equivalence. This last alerts us not to underestimate what might be conceptually entailed in relating equality and equivalence in a way that also clarifies distinctions between them. Indeed, it points to the possibility that those entailments are not insignificant for all students and it reminds us that instruction needs to be sensitive to that possibility.

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