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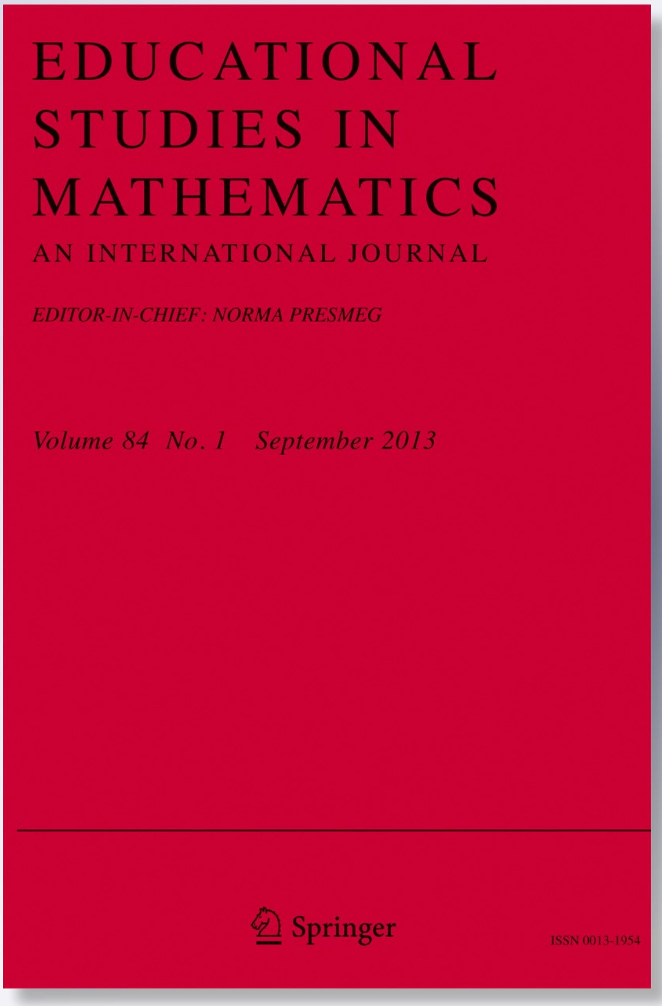
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Articulating syntactic and numeric perspectives on equivalence: the case of rational expressions

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Abstract Our study concerns the conceptual mathematical knowledge that emerges during the resolution of tasks on the equivalence of polynomial and rational algebraic expressions, by using CAS and paper-and-pencil techniques. The theoretical framework we adopt is the Anthropological Theory of Didactics (Chevallard 19:221–266, 1999), in combination with semiotic aspects from the instrumental approach to tool use. The analysis we present is based on interviews carried out with a 10th grade student who participated in our research. Our findings highlight the mathematical knowledge (*technological discourse*) constructed in the process of confronting, differentiating, and articulating the several mathematical techniques and theoretical ideas (pertaining to the *numeric perspective* and the *syntactic perspective* on algebraic equivalence) related to the designed equivalence tasks.

Keywords Algebra · CAS technology · Rational expressions · Syntactic versus numeric perspectives on equivalence · Task-technique-technology-theory framework · Theoretical thinking in algebra

> *The distinctions between polynomial forms and functions tend to be ignored in school mathematics* (Cuoco, 2002, p. 297).

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1 Introduction

The many evolutionary changes occurring within school algebra curricula around the world over the last few decades (see, e.g., Sutherland, 2002) have included a significant shift toward functions within the study of polynomial algebra. Although the algebraic properties that govern the manipulation of symbols in expressions and equations are still emphasized, work with polynomials has fused with what has become the functional leitmotif of school algebra. However, as reflected in the opening words of Cuoco (2002), this fusion tends to ignore the distinctions between the two. This question of the articulation of polynomial forms and functions is but one that is related to the broader issue of the development of hybrid versions of programs of study that attempt to include elements of both traditional (involving expressions and equations) and functional orientations to school algebra. As pointed out by Chazan and Yerushalmy (2003), students become confused regarding distinctions between equations and functions, not being able to sort out, for example, how equivalence of equations is different from equivalence of functions. According to Chazan and Yerushalmy, it is not the case that such questions are unanswerable, but rather that combining functional approaches with more standard treatments of school algebra leads to such dilemmas; and that only rarely are opportunities provided for students to inquire into these questions and to attempt to resolve them.

In considering polynomials in one variable with real coefficients, Cuoco (2002) distinguishes between polynomial functions and polynomial forms as follows. Polynomial functions involve thinking about the letter in a polynomial as a variable and about the polynomial as an input–output machine that can yield a table or a graph, and has all the attributes of real-valued functions of a real variable. In contrast, polynomial forms are viewed as formal expressions with the letter considered as an indeterminate and which involve operations such as factoring, adding, multiplying, and so on. These distinctions are particularly important when students use computer algebra system[1] (CAS) technology because the polynomial-form perspective underlies CAS technology, even if CAS also deals with polynomials as functions.

In their research on equivalence of algebraic expressions, Cerulli and Mariotti (2002) make similar distinctions between what they refer to as functional and axiomatic definitions of equivalence (corresponding to polynomial functions and polynomial forms in Cuoco's (2002) remarks). Their functional definition of equivalence reads: "Two expressions are equivalent if and only if, for all numbers that are substituted into the letters, the two numeric expressions thus obtained give the same result" (p. 161); the axiomatic definition: "If one expression can be transformed into another by using the properties of addition and multiplication, then the two expressions are equivalent" (p. 161). However, they also point out that since, for polynomials in n variables, the functional and the axiomatic definitions are equivalent, they do not go into the particularities of the equivalence of these definitions with their learners. They instead draw upon the functional definition in order to indicate non-equivalence (by means of a numeric counter-example) and the axiomatic definition (by means of properties and theorems) for proving equivalence. While mathematically sound, the use of these two definitions for two different

[1] WIKIPEDIA defines a Computer Algebra System (CAS) as a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form. The expressions manipulated by the CAS typically include polynomials in multiple variables; standard functions of expressions (sine, exponential, etc.); various special functions (Γ, ζ, erf, Bessel functions, etc.); arbitrary functions of expressions; optimization; derivatives, integrals, simplifications, sums, and products of expressions; truncated series with expressions as coefficients, matrices of expressions, and so on. Numeric domains supported typically include real, complex, interval, rational, and algebraic.

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aspects of equivalence comparisons does little to enhance students' awareness that "form and function represent two different ways to think about polynomials" (Cuoco, 2002, p. 296).

One of the dilemmas inherent to Cuoco's (2002) implicit challenge for school algebra is that it is very difficult for students to think about differences between form and function if one remains within the polynomials. A resolution of this apparent impasse is achieved by considering a mathematical arena where the two perspectives of form and function collide—one, for example, that involves a comparison of polynomial and rational expressions for equivalence. The present article deals with exactly such a consideration.

The article begins with a review of the literature that includes past studies related to equivalence of algebraic expressions and the role of technological environments in this area of mathematics. This is followed by a description of the study's theoretical framework, which is based on Chevallard's (1999) Anthropological Theory of Didactics and embedded within the instrumental approach to tool use. We then present a discussion of the two mathematical perspectives regarding equivalence and the related cognitive issues that underlie the study. The bulk of the article is devoted to the analysis of a 15-year-old Grade 10 student's foray into the equivalence of polynomial and rational expressions, when he faces the dilemmas posed by the dialectic between the syntactic and numeric perspectives within a CAS environment. A discussion of the central issues informed by this research concludes the article.

2 Past research on equivalence of algebraic expressions

While a substantial amount of research has been carried out with respect to the equivalence of algebraic expressions, very little of it has dealt explicitly with either the comparison of polynomial and rational expressions or the bridging of syntactic and numeric perspectives. Much of the existing body of research has highlighted the difficulties that students encounter with understanding algebraic equivalence, as well as the importance of being able to work flexibly with algebraic expressions in various forms and of recognizing equivalent expressions (e.g., Arcavi, 1994; Ball et al., 2003; Goldenberg, 2003; Kieran, 1984; Kirshner, 2001; Nicaud et al., 2004; Sackur et al., 1997; Steinberg et al., 1990). For instance, Nicaud et al. (2004) have foregrounded the importance of equivalence, framing it as "a major reasoning mode in algebra, which consists of searching for the solution of a problem by replacing the algebraic expressions of the problem by equivalent expressions" (pp. 171–172). Sackur et al. (1997) have proposed that understanding two algebraic expressions to be equivalent entails knowing that they denote the same numerical value for a given common replacement value and realizing that the usual algebraic transformations performed on them conserve this *denotation*.[2]

Arzarello et al. (2001), in their theoretical analyses of the meaning of symbolic expressions in algebra, have pointed out:

> All possible senses of an expression constitute its so called intensional aspects, while its denotation within a universe represents its so called extensional aspect. … The official semantics used in mathematics, and particularly in algebra, cuts off all intensional aspects, insofar as it is based on the assumption of the extensionality

[2] The term *denotation* is drawn from the work of the renowned mathematician and logician, Frege, who distinguished sense and denotation. For example, the expressions $4x+2$ and $2(2x+1)$ would have different senses, but denote the same functional object. According to Arzarello et al. (1994), "the 'denotation' of a symbolic expression in algebra refers to the number set that is represented by the expression; it is determined by the symbolic expression and by the universe in which the expression is considered (for example the equation $x^2=-1$ denotes the empty set when it is considered in **R** and the set $\{+\mathrm{i}, -\mathrm{i}\}$ when considered in **C**)" (p. 42).

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axiom (two sets are equal if they contain the same elements, independently from the way they are described or produced). (p. 64)

This position regarding the "official semantics" according to which two sets are considered equal will be taken up in the later section, "Mathematical theory of equivalence according to the two perspectives of the syntactic and the numeric."

Suffice it to say here that semantics (the study of meaning) is often contrasted in the mathematical literature with syntax or syntactics (the study of rules governing the behavior of systems, without reference to meaning). More typical distinctions between semantic and syntactic, which associate meaning-making with the semantic and meaning-free manipulation with the syntactic, tend to associate most transformative manipulations in algebra with the syntactic. However, such a dichotomy would not be appropriate in the context of the present study—a study in which all reflective activity involving mathematical objects and their transformations is intimately related to meaning-making. The recent work of our research team on the co-emergence of theoretical ideas within the technical activity of algebra is a case in point (e.g., Kieran et al., 2006). In addition, Kaput (1989) has pointed out that, while "the syntactic/semantic distinction is meant to delineate polar extremes, most symbol-use acts involve a mixture of the two" (p. 175); as well, Thom (1973) has asserted that, in practice, for the mathematician every statement that he/she is working with has some meaning regardless of how formally it is presented. Moreover, Booth (1989) has argued that "our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not. These structural properties constitute the *semantic* aspects of algebra" (pp. 57–58). Indeed, the point is that meaning can be drawn from a variety of sources, including the connections among the forms of algebra, its equivalences, and its property-based manipulation activity (e.g., Cerulli & Mariotti, 2001).

Another body of research dealing with equivalence in algebra includes those studies that have explored the use of computer algebra systems (CAS) in the learning of mathematics at the high school and college levels (e.g., Artigue, 2002; Guin & Trouche, 1999; Lagrange, 2000). For example, Artigue (2002) has drawn on students' CAS work involving the passage from one given form of algebraic expression to another to illustrate that, "equivalence problems arise which go far beyond what is usual in the classroom" (p. 265), asserting that the CAS pushes students to confront issues of equivalence and simplification in ways that are not so easily achieved in more traditional, paper-and-pencil, treatments. More generally, French didactic researchers, who have largely been responsible for elaborating the corpus of research involving CAS, have argued that CAS can be used as a tool to promote the co-development of both technique and theory. More will be said about this research-supported point of view in the next section where we present the theoretical framework of our study.

The research with digital tools by the French *didactique* school has given rise to a number of related studies at the international level on the use of CAS in the teaching of mathematics. Inspired by, and wishing to build upon, the research emanating from France, our Montreal-based research team developed a program of research (e.g., Hitt & Kieran, 2009; Kieran, in press; Kieran & Damboise, 2007; Kieran & Guzman, 2010; Kieran et al., 2008; Kieran & Saldanha, 2008) that has investigated the learning of the technical and theoretical aspects of various topics in high school algebra within CAS environments, including the topic of equivalence. Pivotal to the current article is a study, reported by Kieran et al. (2006), where data from two Grade 10 classes were analyzed with a focus on the ways in which students were beginning to think about equivalence within a context involving polynomial and rational expressions:

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> Students linked the notion of restrictions[3] [within the rational expressions] to the numerical view on equivalence, which makes sense. … Still, individual students struggled with the restrictions. … In all, we noticed that the notion of restrictions in relation to equivalence was not easy to grasp. The confusion was evoked by the tasks, which involved expressions with restrictions [within the rational expressions], by the definition of equivalence, which spoke about the set of admissible values, and by the fact that the CAS techniques neglect the restrictions. In particular, the Test of Equality technique was confusing; if there were restrictions, it provided 'true', whereas the numerical substitution of the restriction provided 'false'. (pp. 227–231)

Many crucial questions regarding the articulation of the syntactic and numeric perspectives within the study of the equivalence of polynomial and rational expressions could not be answered by the above classroom-based study, precisely because it was classroom based and did not permit analyses of a fine-grained nature. Thus, additional analyses of out-of-class student-interview data involving the same tasks were carried out. The analysis presented in this article focuses on the interviews that were carried out with one of the Grade 10 students, one who was quite reflective and who was able to share his reflections with the interviewer. The present analysis distinguishes itself from the classroom study of Kieran and Drijvers by its attention, first, to the cognitive complexity of individual knowledge construction with respect to polynomial-and-rational-expression equivalence; second, to a detailed description of the mathematical theory underlying the two perspectives of the syntactic (i.e., form) and the numeric (i.e., function) within this conceptual area; and third, to the explicit representation of specific elements of the student's meaning-making in relation to the technical and the technological-theoretical of Chevallard's (1999) Anthropological Theory of Didactics.

3 Theoretical framework of the study

For our theoretical framework, we draw principally from Chevallard's (1999) Anthropological Theory of Didactics (ATD). While being a theory in its own right, the ATD has also been integrated within the instrumental approach to tool use (see Monaghan, 2007, for a discussion of the two main currents within the instrumental approach). Thus, the integration of the ATD within the instrumental approach to tool use allows us to draw upon and elaborate the semiotic affordances of the latter frame while maintaining salient aspects of the former.

The development and use of digital tools such as CAS (e.g., Artigue, 2002; Drijvers & Trouche, 2008; Lagrange, 2000) have led to altering the way we think about the relation between technical and conceptual knowledge in algebra education. Central to this rethinking has been the theoretical elaboration of a framework whose generation and use within studies involving these tools is a reflection of both the work of Vygotsky (1930/1985) on tools in general and on cognitive tools in particular, and to a lesser extent that of Piaget on cognition and abstraction. This framework, referred to as the instrumental approach to tool use, is one that encompasses elements from both cognitive ergonomics (Rabardel, 2002; Verillon & Rabardel, 1995) and the Anthropological Theory of Didactics (Chevallard, 1999).

[3] The set of restrictions of a rational function $f(x) = \frac{p(x)}{q(x)}$ (p and q polynomials in x) of real variables with real values is $A = \{x \in R : q(x) = 0\}$. The function f cannot be evaluated on the elements of the set A; its domain is $\mathbf{R}-A$.

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The instrumental approach to tool use makes a distinction between artifact and instrument. While *artifact* is the term used to refer to the physical tool itself, *instrument* is used to refer to the tool in conjunction with the cognitive schemes that the user develops while using the tool. The process by which the physical tool becomes a cognitive instrument is called instrumental genesis and is said to have two components: *instrumentalization* whereby the user's background knowledge and experience shape the way the tool is being used and *instrumentation* whereby the affordances of the tool shape the way the user's thinking develops (see Artigue, 2002).

Within Chevallard's (1999) ATD, the objects of mathematical knowledge emerge from didactic institutional practices consisting of four components: the *types of tasks* in which the objects of knowledge are immersed; the *techniques* or ways of solution of these tasks; the discourse that explains and justifies the techniques, named *technology*; and the *theory* that provides the "abstract and generative" (p. 228) basis for the technological discourse.

In their integration of the ATD into the theoretical frame of the instrumental approach, Artigue and her collaborators (see, e.g., Artigue, 2002; Lagrange, 2002) collapsed the four ATD components into three: task, technique, and theory so as to reserve the term *technology* for the digital tools being used in their studies. Vital to the line of research and theoretical thinking engaged in by Artigue and her collaborators in their use of the instrumental approach has been the claim, supported by their research findings, that a technique has not only a pragmatic but also an epistemic value:

> Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration; it also serves as an object for conceptual reflection when compared with other techniques and when discussed with regard to its consistency. (Lagrange, 2003, p. 271)

It is precisely this interaction between technique and theory, as constituted by the learner, which we wish to explore more deeply in this study. To do this, we have decided to use the original version of the ATD with its four components of task, technique, technology, and theory, rather than the abbreviated version as adapted by Artigue and her collaborators. This choice will allow us to juxtapose the discourse that explains and justifies the techniques (i.e., the *technology* during its constitution by the student) with the mathematical *theory* that provides the institutional basis of the technological discourse, in other words, to discuss the relation between the technical and the technological-theoretical in the student's meaning-making against the more formal mathematical subtext of the given equivalence tasks. More broadly, the instrumental approach to tool use permits, at the same time, an analysis of the semiotic role played by the digital tool in the co-emergent constitution of the technical-theoretical knowledge.

Radford (2006), in his semiotic-cultural approach to analyses of students' meaning-making, has elaborated on the way in which, through words, artifacts, and mathematical signs—which he refers to as semiotic means of objectification—the mathematical object is made apparent to the student in an "objectification process in the course of which the student's subjective meanings are refined" (p. 57). Digital tools, such as CAS, are examples of such semiotic artifacts. The CAS tool with its own internal logic carries out "simplifications" and displays the results in a form that is not necessarily what students would expect, having been endowed with a theoretical content prior to the students' mathematical experience. Reflecting on such CAS output and trying to reconcile it with, and integrate it into, their existing mathematical knowledge is constitutive of the powerful semiotic role that such digital artifacts can play. When the design of the tasks takes advantage of this potential of CAS tools, the mathematical concepts (both technical and theoretical) embodied within the tasks and tools can be transformed into objects of consciousness for students. Radford has described the nature of the objectification process by which the student's subjective meanings are refined as follows:

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On one side, meaning is a *subjective* construct: it is the subjective content as intended by the individual's intentions. Meaning here is linked to the individual's most intimate personal history and experience; it conveys that which makes the individual unique and singular. On the other side and at the same time, meaning is also a *cultural* construct in that, prior to the subjective experience, the intended object of the individual's intention (*l'objet visé*) has been endowed with cultural values and theoretical content that are reflected and refracted in the semiotic means to attend to it. … It is in the realm of meaning that the essential union of person and culture, and of knowing and knowledge are realized. (pp. 53–54)

In the following mathematical discussion of the theoretical content inherent in syntactic and numeric perspectives on equivalence, we foreshadow the complexity of their articulation in the process of meaning-making engaged in by the student.

4 Mathematical theory of equivalence according to the two perspectives of the syntactic and the numeric

In our opening remarks, we briefly situated our study in relation to Cuoco's (2002) distinction between polynomial form with its syntactic underpinnings and polynomial function with its numeric underpinnings. In this section, we elaborate more fully on the nature of the mathematical objects that are typically referred to in high school algebra as polynomial/rational expressions[4] and polynomial/rational functions. This elaboration will serve as background for the later discussion of student meaning-making within this study.

Algebraic expressions are ordered series of symbols (numbers, parentheses, arithmetic operator, symbol for a variable) conforming to a given grammar (see, e.g., Kirshner, 2001). The algebraic expressions that we consider in this study can correspond either to rational fractions or to rational functions, depending on the way in which they are interpreted. Let us begin with *rational fractions*[5] in one indeterminate X with coefficients on the set of real numbers $\mathbf{R}$, that is, elements of the field $\mathbf{R}(X)$. If we see algebraic expressions as denoting elements of the field of rational fractions $\mathbf{R}(X)$ and we center our attention on their syntactic-algebraic properties, then we say that two expressions F and G are *equivalent from the syntactic perspective* when they have a *common algebraic rewriting.*[6] This rewriting can be obtained, for instance, by applying the algebraic properties of the field $\mathbf{R}(X)$, such as the commutative, associative, distributive, or the identity properties, but also the properties associated with the theorems of factoring, canceling, long division, etcetera.

For example, expressions $G(X)=\frac{1}{X}$ and $H(X)=\frac{X-2}{X^2-2X}$, when considered as rational fractions, are equivalent from the syntactic perspective because expression H can be

[4] The definitions that will follow are given for rational fractions, considering that the ring of polynomials $\mathbf{R}[X]$ is *embedded* in the field of rational fractions $\mathbf{R}(X)$. The same is done for rational functions and polynomial functions.

[5] The *field of rational fractions* in one indeterminate $\mathbf{R}(X)$ is the field of fractions of the polynomial ring in one indeterminate $\mathbf{R}[X]$. See, for example, Grillet (2007).

[6] The definition of syntactical equivalence can also be given in the following way: Two expressions are equivalent if their *cross products* are equal. For example, $G(X)=\frac{1}{X}$ and $H(X)=\frac{X-2}{X^2-2X}$ are syntactically equivalent because their cross products are equal: $(1)(X^2-2X)=(X)(X-2)$.

This is a more formal definition from the mathematical point of view, which follows the formal construction of the field $\mathbf{R}(X)$, but is not sensitive to the multiple variants to be considered in a student's productions.

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rewritten as: $\frac{1}{X} \cdot \frac{X-2}{X-2}$, and *canceling* the common factors of the denominator and the numerator,[7] we obtain that H is *rewritten* as: $1 \cdot \frac{1}{X} = \frac{1}{X} = G(X)$.

On the other hand, we can consider algebraic expressions as denoting *rational functions*[8] by substituting a number into the indeterminate symbol X. In this case, we will say that two algebraic expressions f and g are *equivalent from the numeric perspective* when $f(x) = g(x)$ for all x in the *common domain* $\mathbf{R}-F$ (F finite set).[9] For instance, expressions $g(x) = \frac{1}{x}$ and $h(x) = \frac{x-2}{x^2-2x}$ are equivalent from the numeric perspective in the *common domain* $\mathbf{R} - \{0, 2\}$.

Although the notions of rational fractions and rational functions *concur*[10] when the domain of validity is **R**, the two perspectives of equivalence emphasize different aspects. For the numeric perspective, it is essential to include the study of the characteristics of domains and images for the corresponding rational functions being compared. This differentiates the numeric perspective from the syntactic one, which does not require taking into account the domains and images of the expressions. Grappling with and coordinating these different aspects lies at the heart of students' meaning-making for expressions and their equivalence.

For the syntactic perspective, the rewriting of expressions plays a central role. On the basis of the algebraic properties of the operations of the field of rational fractions **R**(X), the rewriting of expressions allows for verifying equivalence and for obtaining equivalent expressions. These operational characteristics determine the syntactic perspective. To use another example, the expressions $M(X) = X^2 + X - 6$ and $N(X) = \frac{X^3-3X^2-10X+24}{X-4}$, when viewed as rational fractions and invoking the syntactic perspective, are equivalent because the syntactic rewriting of the latter rational fraction yields the former, which is the same rational fraction. No consideration of domain is involved when viewing an expression as a rational fraction.

While the rational expressions $G(X) = \frac{1}{X}$ and $H(X) = \frac{X-2}{X^2-2X}$ denote the same rational fraction, they do not denote the same rational function. On the one hand, these expressions are numerically equivalent, subject to certain restrictions. They have the same values, except when x=0, 2. On the other hand, these expressions can be re-written as the same expression by using the algebraic properties of the field of rational fractions **R**(X), that is, they are *syntactically equivalent*. Although two expressions may be syntactically equivalent because one expression can be transformed into the other following certain syntactic rules, the substitution of a number into the variable symbol of two syntactically equivalent expressions may not yield the same numerical result for each expression. This is explained by the fact that two particular transformations of expressions, cancelation and expansion, are special in the sense that equivalence

[7] For rewriting $H(X)$, one could also perform the division indicated by: $\frac{X-2}{X-2} = (X-2) \div (X-2) = 1$, when $H(X)$ is considered to represent a quotient of polynomials.

[8] By substituting a number into the indeterminate symbol of a rational fraction, we obtain arithmetic expressions that can be simplified into a number or into impossibility (1/0). This allows considering the corresponding function of the rational fraction $f(X)$, that we will call the *rational function*; that is to say a set of ordered pairs (x, $f(x)$) where x is in a set of values whose substitution gives a number and $f(x)$ the number obtained by substitution.

[9] This definition allows us to have a well-defined *transitive property* of the equivalence relationship: If f and g are equivalent on $\mathbf{R}-F_1$ and g and h are equivalent on $\mathbf{R}-F_2$, then f and h are equivalent on $\mathbf{R} - (F_1 \cup F_2)$ (for F_1 and F_2 finite sets).

[10] As do polynomials **R**[X] and polynomial functions **R**[x]. When the coefficients are taken in the field of real numbers **R**, for every polynomial function there exists one unique polynomial, and vice versa. But this is not always true. If the coefficients of the polynomials are in a field of *characteristic* different from zero, different polynomials may give rise to the same polynomial function. For example, the polynomial $X^2 - X \in \mathbf{Z}_2[X]$ corresponds to the zero function.

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Table 1 The three expressions that are central to this study

Expression 2:	$(x^2+x-20)(3x^2+2x-1)$
Expression 3:	$(3x-1)(x^2-x-2)(x+5)$
Expression 5:	$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$

obtained through these transformations does not guarantee that if a number substituted into one expression gives a number, then it will also be the case for the other. Whether or not two expressions syntactically equivalent through one of these transformations denote the same mathematical entity depends on the mathematical entities being considered. In the case of rational fraction, the answer is yes, and in the case of rational function, the answer can be yes or no. In other words, when considering equivalence, for rational fractions, a syntactic perspective is sufficient; for rational functions, it is not—a numeric perspective is also required. Thus, the designed tasks, which involved both polynomial and rational expressions, faced students with the necessity of articulating the differences between the two perspectives on algebraic equivalence. However, the tasks did not aim at developing an understanding of the two mathematical objects at play, that is, rational fraction and rational function. Rather, it was the articulation of the two perspectives on equivalence, the syntactic and the numeric, in the face of polynomial and rational expressions that was our main aim.

The underlying research question of this study is the following: *What is the nature of the technological discourse that students develop in the process of articulating the syntactic (form) and numeric (functional) perspectives when grappling with the equivalence of polynomial and rational expressions in a task-based, CAS and paper-and-pencil, environment?*

5 Methodological considerations of the study

The instrumental approach, with its focus on tools and tool use, in concert with the ATD and its emphasis on the epistemic role played by technique in the development of theoretical knowledge related to given tasks, suggested a methodology where consideration had to be accorded to both task design and the obtaining of detailed cognitive data, as well as to the nature of the analysis to be carried out, with particular attention to the role of the CAS tool. This section briefly describes these considerations, first, those related to the design of the tasks and, second, those pertaining to the research participants and the methods of data collection and analysis.

5.1 The design of the tasks

The research team developed a set of task sequences on equivalence, involving polynomial and rational expressions, within an environment that included both paper-and-pencil and a CAS tool (the TI-92 Plus handheld calculator). While the full set of tasks was used in the classroom study (see Kieran et al., 2006), the research featured in this article involved a subset consisting of four task sequences from the initial set of activities used in the classroom study, those revolving around the articulation between the syntactic and numeric perspectives on equivalence (see the Appendix for the task questions included in this study).[11] Table 1 displays the three expressions that were the focus of this study.

The expressions and the tasks in which they were embedded were designed with several considerations in mind. First, the expressions were of a level of complexity

[11] See the project web site for the entire set of activities: http://www.math.uqam.ca/APTE/TachesA.html

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that would not permit students to be able to tell just by looking at them whether or not they were equivalent. Second, the expressions included both a polynomial and a rational expression that were syntactically equivalent, as well as a non-equivalent polynomial expression. More specifically, rational Expression 5 can be simplified via factoring and cancelation, or expansion and long division, to the polynomial Expression 3; Expression 2, while having some factors in common with both Expressions 3 and 5, is not equivalent to them. Third, the task sequences comprised questions of both a technical and a theoretical nature so as to encourage the co-emergence of both technical and theoretical knowledge. Fourth, the task questions, which were at times paper-and-pencil oriented, and at other times CAS oriented, were sequenced in such a way as to allow students who had no prior experience with formal concepts of equivalence to come gradually to an understanding of algebraic equivalence in terms of both numeric and syntactic perspectives.

The first task sequence involved comparing expressions by numerical evaluation. The CAS technique of numerical substitution was thereby to be introduced right from the outset. Using the CAS tool to do their computations, students were to substitute two given numbers into the given expressions, followed by two other numbers of their own choosing. The aim was to have them notice that while some values for x yielded the same result for the expressions, this was not always the case. The open-ended questions of a theoretical nature were to allow for the expression of such noticing. The students were then to be asked to conjecture what would happen if they extended the table to include other values of x. By asking this question, we were indirectly inquiring as to whether students were beginning to think about the numerical restriction on Expression 5, and by extension whether they might express the idea that not all numerical substitutions were guaranteed to give the same results for two given expressions.

The second task sequence involved comparing expressions by means of paper-and-pencil algebraic manipulation, namely by factoring. This sequence aimed first at having students make a conjecture based on their prior numerical work as to which of the given expressions might be re-written in a common form. After factoring the given expressions, they were then to be asked to explain in what way these algebraic manipulations supported (or not) each of those conjectures. Technological-theoretical questions of this sort were designed to encourage the beginnings of a reflection on the articulation of the two perspectives on equivalence. In addition, the factored form of the given expressions was such that students might notice that certain expressions involved the same factors (i.e., Expressions 3 and 5, after the cancelation of the common factor in numerator and denominator of Expression 5), whereas others (i.e., Expression 2) might share only some factors with the other expressions. The developing of this awareness was considered to be potentially useful for the follow-up task sequence that was to involve working with the expanded forms of the three given expressions; more specifically, the combination of factoring and canceling could be seen as being equivalent operation-wise to dividing the expanded form of the numerator by the factor that constituted the denominator.

The third task sequence focused on the use of CAS to obtain the expanded forms of the given polynomial and rational expressions. While the expanded form of Expression 5 by CAS would yield the same polynomial form as for Expression 3, no information would be provided by the CAS tool as to domain restrictions regarding Expression 5, or to the fact that the simplified form of Expression 5 involved a domain change. In other words, in applying the CAS techniques of expansion, factoring, and automatic simplification, the polynomial

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and rational expressions are treated by the CAS as rational fractions. It was intended by the designers that this characteristic of the CAS could help reinforce students' own paper-and pencil results with respect to the factored and simplified form for Expression 5, which was thus syntactically equivalent to Expression 3.

The fourth task sequence was to introduce the CAS technique that we refer to as the CAS Equivalence Test. This test involved entering two given expressions into the CAS with an equal sign between them, and then pressing the Enter button. Students were first to be asked to work with Expressions 3 and 5: $(3x-1)(x^2-x-2)(x+5)=\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$. The CAS would then display "true" as its output, which students would be asked to interpret. Next, we were to introduce the CAS Test of Numeric Equality, which would involve, in this case, replacing x by -2 in the above equation and interpreting the result given by the CAS. The fact that the CAS would then display "false" was intended to be a pivotal moment within the activity. In applying the CAS Equivalence Test, polynomial and rational expressions are treated by the CAS as rational fractions, whereas in applying the CAS Test of Numeric Equality the same expressions are treated as rational functions. It was considered that this particular task would serve as a point of confrontation between the two different perspectives on equivalence and would have the potential to lead students to attempt to explain and to justify the two seemingly contradictory results yielded by the CAS. In addition, the same CAS Test of Equivalence, when supplied with Expressions 2 and 3 as input, would return the given equation as its output (this is the CAS's manner of displaying that the given expressions on each side of the equal sign are not syntactically equivalent). While students might expect an output of "false" in this case, they would be mistaken because there are certain values of x for which both sides of the equation would yield the same result. This aspect of the task was intended to encourage the related idea that syntactically non-equivalent expressions may be equal for certain values and that these values are precisely those that are the solutions of the equation formed from such a pair of expressions.

5.2 Methods: the participant and the data collection

The 10th grade student who participated in this study was one who volunteered to spend an hour or so after class each day for a few days, working on the designed tasks and being interviewed while he was engaging in these tasks. His teacher described him as a very good student who reflected on his mathematics and who posed interesting questions during his mathematics classes. The interviews, conducted by a member of the research team, were carried out 2 weeks before the same tasks were presented within the classroom study. The student, Andrew, received a modest stipend for his participation.

All interviews were videotaped and later transcribed. Andrew's CAS calculator was hooked up to a view-screen that projected onto the wall behind him, thus allowing both the interviewer and the videographer to take note of, and to film, everything that Andrew entered into the CAS device. Andrew was also able to look back at the view-screen projection to see what the researchers were filming. Andrew's prior experience with the CAS tool was quite basic, but adequate. He had been introduced to the CAS tool and to some of its commands during the week preceding the interviews when he and his classmates had worked on an activity involving factoring the sum and difference of cubes. At that time, they had become acquainted with the CAS commands: Factor and Expand. The CAS Evaluation command and the Equivalence and Numeric Equality Tests were introduced during the interview itself.

As was the case with his classmates, Andrew had already learned the four basic operations with polynomials, and paper-and-pencil techniques for factoring certain binomials and trinomials, and for solving linear and quadratic equations, during his 9th grade

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mathematics course the year prior. The pretest that was administered to the class at the outset of the larger classroom study showed that Andrew was quite skilled in the syntax of symbol manipulation of polynomial expressions. However, as per the Quebec school mathematics curriculum, he had not yet had any formal school experience with the notion of equivalence, nor with rational algebraic expressions. During the usual mathematics lessons of the previous month, the class had begun the textbook treatment of functions, which included the standard introductory topics of domain and range, dependence, relation versus function, and modes of representation, along with graphing calculator technology. The interviews with Andrew took place shortly after this classroom work with functions.

The data that were analyzed included all of the task worksheets on which Andrew wrote his responses and theoretical reflections, as well as the transcripts of the interviews, all of which were conducted while Andrew was working on the designed tasks. It is noted that both Andrew and the interviewer posed questions to each other—Andrew, so as to better understand the intent or meaning of a task question, and the interviewer, so as to better understand Andrew's thinking. Their conversations introduced some unplanned-for task activity.

The approach we took in analyzing the data was, through the solution of the designed tasks, to look for the confronting, comparing, and exploring of the different techniques available to Andrew for judging equivalence—both CAS and paper-and-pencil—to gauge their *epistemic value*, but we were also looking for the mathematical conceptual basis that supported their application. We were interested in the descriptions that the student developed for explaining the differences and "contradictions" obtained by the application of the different techniques. In other words, we were interested in studying the *technological discourse* developed for conciliating the differences between the two perspectives of algebraic equivalence. In line with Radford (2006), we wished to seek indications of the *objectification process* by which the student's subjective meanings on algebraic equivalence were being further developed and refined.

6 Results

The results we present are drawn from an analysis of the interviews with Andrew and correspond to the different task sequences proposed to him, as well as the spontaneous tasks he generated for himself. First, we discuss Andrew's elaboration of a conjecture on the equivalence of the given algebraic expressions. Then, we analyze the way in which he justified his conjecture through factoring and expanding techniques. Later, we present the part of the interview where the differences between the two perspectives on algebraic equivalence are confronted. Finally, we discuss the way in which Andrew made explicit these differences and then reconciled and articulated them. We note that our ATD-inspired representations of the techniques and technological explanations suggested by Andrew's work make use of the more-commonly used polynomial-and-rational-expression terminology, while our accompanying analyses include the more formal terminology of rational fraction and rational function that was introduced in the a priori mathematical analysis.

6.1 Conjecturing the equivalence of algebraic expressions from a numeric perspective

The first part of the interview (*Comparing expressions by numerical evaluation*) consisted of evaluating given algebraic expressions and producing a conjecture regarding their numeric equivalence. In this part of the interview, we introduced *evaluation with CAS* as an efficient technique for obtaining the value of an expression for specific values of x: the

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Table 2 Evaluation of algebraic expressions by using CAS

Expressions	Values of x			
	$\frac{1}{3}$	-5	6	7
Expression 2: $(x^2+x-20)(3x^2+2x-1)$	0	0	2618	5760
Expression 3: $(3x-1)(x^2-x-2)(x+5)$	0	0	5236	9600
Expression 5: $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	0	0	5236	9600

expression is typed into the calculator, followed by the symbol "|" (by using a special key on the calculator), and then inserting the value of x to be used in evaluating the expression. The CAS returns the corresponding value of the expression (for instance, in evaluating: $(x^2+x-20)(3x^2+2x-1)\big|x=\frac{1}{3}$, the calculator returns 0 as the result). Table 2 shows the expressions that were evaluated in the interview, the values of x used for their evaluation, and the results obtained. In the interview, two values of x were given: $\frac{1}{3}$ and −5; we asked Andrew to choose two more values. He chose 6 and 7.

While Andrew filled out the table of results, he spontaneously identified several numerical relationships, in particular the equality of values of the expressions for several values of x that had been substituted. Once the table was completed, the interviewer asked:

> I: What do you observe about some of those results?
> A: Three and five are the same results, with… no matter what number you use, with no matter what x you use. The result for expressions two, three and five using the given values are equal to zero, for the given values, all are equal to zero. And the results for three and five, using the chosen values of 6 and 7, are equal to each other.

The numeric work of these activities served as an empirical basis for conjecturing the nature of the values that the expressions could take for any possible value of x. The next task question of the activity was as follows: *Based on your observations with regard to the results in the table above, what do you conjecture would happen if you extended the table to include other values of x?* Andrew wrote on his answer sheet: *The results for expressions 3, 5 would continue to be equal to each other.* In terms of our theoretical distinctions between the two perspectives on equivalence, we say that Andrew was conjecturing with respect to the *numeric equivalence* of Expressions 3 and 5, considering them as rational functions—even if Andrew never used such terminology.

It also seems important to remark that, throughout the interview, Andrew often resorted to the numeric approach for checking his work. For instance, in moments of doubt or conflict with

Table 3 CAS evaluation for conjecturing the numeric equivalence of algebraic expressions

Technique	If the expressions are not identical, evaluate them for some[a] values of x. If the expressions are identical, there is no need to evaluate.
Technology	If two algebraic expressions take on the same values for a set of values of x, they can take on the same values for any value of x, that is, they can be *numerically equivalent* expressions.

Type of task: to conjecture numeric equivalence of two given algebraic expressions

[a] The problem associated with the exact number of values necessary for establishing the numeric equivalence of two expressions is generally taken up by students in their later algebra courses. Thus, Andrew had not yet studied the problem of determining a polynomial function of degree n from knowing $n+1$ different values of its image set.

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respect to the equivalence of expressions, he verified his syntactically obtained answers by evaluating the involved algebraic expressions for several values. CAS played a central role for these verification tasks, reassuring Andrew that his decisions as to equivalence were well founded and contributing to his understanding of the connections between his syntactic and numeric work—thereby illustrating the *epistemic value* of the CAS evaluation technique.

Table 3 presents our analysis of Andrew's conjecture on the numeric equivalence of Expressions 3 and 5, in terms of the *technique* and the *technology* involved, as per Chevallard's (1999) ATD.

6.2 Two syntactic techniques for gaining certainty: expanding and factoring

In the next segment of the interview, the interviewer asked Andrew to justify his conjecture on the numeric equivalence of Expressions 3 and 5 for all numbers. He asked:

> I: What if somebody else said: "well, you can't try all the numbers". What way would you try to prove to them that the results for those would always be equal, without trying all the numbers?
> A: I'd simplify the expressions or I'd factor them out, and then or, and then I'd just, basically my equation would be expression for number 3 equals expression for number 5. I would see if that's correct.

As shown below, Andrew expanded or factored the given expressions so as to obtain certain "forms" that he could compare. According to Andrew, if two expressions could be rewritten to yield the same form, then they would take on the same values for any x, in other words, they would be the same rational function. In this way, Andrew resorted to syntactic techniques for justifying the numeric equivalence of these expressions.

We next present our analysis of Andrew's justification of his conjecture regarding the numeric equivalence of Expressions 3 and 5: $(3x-1)(x^2-x-2)(x+5)$ and $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$.

6.2.1 Supporting numeric equivalence by means of a technique from the syntactic perspective: the expanding technique

Andrew spontaneously justified his conjecture by *expanding* and *simplifying* the given expressions. With paper and pencil, he worked out the indicated operations (multiplications and divisions) and simplified similar terms, considering the expressions as denoting a polynomial and an indicated quotient of two polynomials. In this way, he obtained a form of the given expressions by means of which he could compare them: the *expanded form*.

For Expression 3, Andrew multiplied and, by simplifying similar terms, obtained the polynomial:

$$3x^4+11x^3-25x^2-23x+10$$

Later, for Expression 5, he explained to the interviewer how to simplify by performing long division of the two polynomials of this rational expression, obtaining a polynomial expression as result.

> A: I would have figured it out using division, and then if your result is equal to this answer [expanded form of Expression 3], then the same will be true… that they're equal to each other.

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Table 4 Expanding technique for establishing syntactic equivalence of algebraic expressions

	Polynomials	Rational expressions
Technique	Expand the expressions and simplify similar terms. Compare the polynomials obtained.	Perform the long division corresponding to the rational expression (dividing the numerator polynomial by the denominator polynomial). Compare the polynomial quotients obtained.
Technology	Every polynomial expression can be expanded and simplified by performing the indicated operations (additions, subtractions and multiplications). Given two polynomial expressions, they are syntactically equivalent if the same polynomial is obtained as a result of expanding and simplifying them.	Given a rational expression f/g, it is always possible to perform the corresponding division to obtain polynomials q and r such that $f=qg+r$. Given two rational expressions, they are syntactically equivalent if the same polynomial quotient is obtained as a result of performing the corresponding long divisions of numerator by denominator for each expression, provided there are no remainders.

Type of task: establishing syntactic equivalence of two given algebraic expressions (i.e., rewriting them as the same expression)

It is important to remark that, in this division, Andrew did not consider any restrictions on the polynomial divisor. It was a purely syntactic application of the long division of a polynomial by another polynomial in order to obtain a polynomial quotient (in this case the remainder being zero). According to Andrew, if the result of the division were equal to Expression 3, it would be established that Expressions 3 and 5 were one and the same.

Table 4 shows the analysis of the techniques used by Andrew for rewriting the expressions and the (underlying) technology supporting their application. Note that, although polynomial and rational expressions are both considered rational fractions within the syntactic perspective, the collapsing of these two kinds of expressions under the one term of rational fraction does not allow us to differentiate the set of techniques and "technological discourses" involved in Andrew's solutions. Thus, we maintain their separateness and refer to them as polynomial and rational expressions, even if the mathematical object "rational fraction" is more formal and general.

For Andrew, the fact that he obtained the same results when expanding Expressions 3 and 5 was not yet being explained in "technological terms" that embraced both the syntactic and numeric perspectives. The results seemed to be simply a response to the task—a consequence of the application of the techniques that he had for rewriting expressions.

6.2.2 Supporting numeric equivalence by means of another syntactic technique: the factoring technique

After Andrew's spontaneous use of expanding to justify his conjectures as to numeric equivalence, the interview continued with Part II—on comparing expressions by algebraic manipulation, this time with Andrew using the factoring technique for supporting his conjecture about the numeric equivalence of expressions, that is, their equality as rational functions. First, Andrew was asked to make a conjecture as to which of the above set of given expressions (Expressions 2, 3, and 5) might be re-expressed in a common form. Based on his results from the previous part of the

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interview (using the expanded form), Andrew answered that Expressions 3 and 5 could be rewritten as the same expression. In his words:

A: I believe that $3x^4 + 11x^3 - 25x^2 - 23x + 10$ is a common form of expressions 3 and 5.

The interviewer continued by asking him to establish the equivalence without using the expanded form, but still with paper and pencil: *To test your conjecture by means of paper-and-pencil algebra, re-express the given expressions in another form (not the expanded form).*

Andrew knew some basic techniques for factoring polynomials, including *factoring by grouping*. He factored Expressions 2 and 3 and obtained:

$$\text{Expression 2} : (x^2 + x - 20)(3x^2 + 2x - 1) = ((x + 5)(x - 4))((x + 1)(3x - 1))$$
$$\text{Expression 3} : (3x - 1)(x^2 - x - 2)(x + 5) = (3x - 1)((x - 2)(x + 1)(x + 5))$$

For Expression 5, he considered performing the division, but after reviewing the answers that he obtained from the other expressions, he put the polynomial of the numerator into factored form and canceled the factor that was in common with that of the polynomial denominator. Finally, he obtained the following:

$$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)} = \frac{((x-2)(x+5))(3x-1)((x+1)\cancel{(x+2)})}{\cancel{(x+2)}} = (x-2)(x+5)(3x-1)(x+1)$$

He then compared the factorizations:

A: So, some are… there's a lot of similar factors… $x - 2$ is seen in three and five, basically they're all the same, all the factors are the same. Except $x - 4$ in the first one [Expression 2], and in the other two it is $x - 2$ [Expressions 3 and 5]. I don't know.
I: Is that surprising?
A: Uh, yeah, I guess so. Well I wouldn't expect these to be equal [pointing to Expressions 3 and 5], just 'cause I wouldn't have a clue just by looking at them…

It is of interest that Andrew was initially surprised by the fact that, after cancellation, Expressions 3 and 5, had the same factors—because, as he said, their original expressions were so different. Nevertheless, we wonder whether he had ever before thought about the relationship between polynomial long division and the cancelling of factors in the numerator and denominator of a rational expression, which he had just carried out for Expression 5. By means of the syntactic techniques of expanding and factoring, Andrew obtained the same results (polynomial quotient and simplified rational fraction for Expression 5) for Expressions 3 and 5. The factorized forms allowed Andrew to compare the expressions (factor by factor). Based on this comparison, he corroborated the syntactic equivalence of Expressions 3 and 5 and their non-equivalence with Expression 2. As he said:

A: It makes sense that this one [pointing to the factorization of Expression 2] it is a bit different because the results were different. And it makes sense that these [factorizations of Expressions 3 and 5] were equal. That's what I said before.

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Table 5 Factoring technique for establishing syntactic equivalence of algebraic expressions

	Polynomials	Rational expressions
Technique	Factor the polynomials. Compare the factors.	Factor the numerator and denominator polynomials and cancel the common factors. Compare the factors of the simplified rational expressions.
Technology	Some second-degree polynomials can be factorized as the product of two non-constant linear factors. Given two polynomial expressions, they are syntactically equivalent if they have exactly the same factors.	It is possible to obtain a "simplified form" of a given a rational expression *f/g*, by canceling common factors of its numerator and denominator polynomials. Given two rational expressions, they are syntactically equivalent if they have the same simplified forms.

Type of task: establishing syntactic equivalence of two given algebraic expressions (i.e., rewriting them as the same expression), through factoring

Table 5 presents the analysis of the techniques and technologies underlying Andrew's solution of this set of tasks.

By this moment of the interview, Andrew had articulated the equivalence of Expressions 3 and 5 through a combination of several approaches; namely, he had stated:

- "all the factors are the same" [their *factorized forms* have exactly the same factors, after canceling common factors; here the expressions are denoting polynomials and rational expressions];
- "they have the same common form" [identical *expanded form*, after multiplying, simplifying similar terms or dividing the polynomials involved; here the expressions are denoting polynomials and rational expressions];
- "three and five have the same results, with… no matter what number you use, with no matter what x you use" [they take on equal values when being evaluated for any value of x; here the expressions are denoting polynomial functions and rational functions].

In the following section, we present the analysis of Andrew's subsequent strategies for facing the difficulties that arose when considering an aspect of the numeric perspective to which he had not attended until now, but which generated a conflict between the two perspectives on algebraic equivalence: the domain restrictions of the expressions. In the process of resolving this issue, Andrew constructed a new technological discourse for embracing both the numeric and the syntactic perspectives on the equivalence of algebraic expressions.

6.3 The issue of domain constraints and equivalent syntactic forms

The interviewer continued, asking Andrew to find the *domain of definition* for the rational functions given by Expression 3: $(3x-1)(x^2-x-2)(x+5)$, and Expression 5: $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$. The interviewer asked:

I: Is there any value of x that would not be permissible as a replacement value for x in Expressions 3 and 5?

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A: Maybe zero. No, actually everything would work.
I: When you see that denominator $x+2$…
A: Hmm, hmmm… Negative 2. 'Cause then it would be zero.
If x were negative 2… then the denominator would be negative 2 plus 2, which is zero and anything over zero is equal to zero.

Note the prompt by the interviewer to take notice of the denominator, $x+2$. Andrew knew that having a zero in the denominator presented a "special case" for determining the domain for the function represented by Expression 5 (in his regular math class, he had been taught functions, function domains, and function images). At that moment, the interviewer used numeric examples for reminding Andrew that division by zero does not equal zero, but is rather "undefined."

From this intervention, Andrew concluded that the function given by Expression 5 was not defined for $x=-2$. The interviewer continued, asking about the consequences of this non-definition with respect to the equivalence of Expressions 3 and 5:

A: [Andrew reads aloud the task instructions: *In Part I(C) you made some conjectures based on numerical evaluations of expressions. Explain in what way the algebraic manipulations in Part II(B) [involving factoring] supported or not each of those conjectures*.]
Basically, I had said that anything, any x value that would be applied to 3, to Expressions 3 or 5, that they would always be equivalent, they would always equal each other.
But, well, I should have seen this before too. I don't know why I didn't… I know now that negative 2 wouldn't work and I'm sure there are others that wouldn't work either.
I: So, here we're referring to the conjectures you made in Part IC [on the numerical equivalence of the expressions]. So would you say that the algebraic manipulations you did [the interviewer refers to the factorizations that Andrew did] supported your conjectures?
A: Well just, it [he is referring to the factorized form for Expression 5, once the common factors are cancelled: $(x-2)(x+5)(3x-1)(x+1)$] is, like, another form of the expression, which is, I guess, once the expression is factored out, and then they're still equal to each other. So, that makes sense [small laugh]. Basically, Expressions 3 and 5, when they are fully factored, at least to my capability, they're equal to each other, still. So, it just supports my conjecture that, with any x value, excluding negative 2, they would be equal to each other.

Our tasks were designed so that students would confront the differences between syntactic and numeric equivalence. On the one hand, the function given by Expression 5 is not defined for $x=-2$, its domain is $\mathbf{R}-\{-2\}$. On the other hand, the function given by Expression 3 is defined for every real number, that is, its domain is the set of real numbers $\mathbf{R}$. So, these expressions (the functions that they denote) are not numerically equivalent for every real number but rather for the numbers in the *common domain* $\mathbf{R}-\{-2\}$.

In this first contact with the differences between the two perspectives on algebraic equivalence, Andrew incorporated the restriction as an exception to the equality of the values of the expressions: although these expressions are syntactically equivalent (they can be rewritten as the same polynomial expression), there is a value of x for which their values are not equal (they do not denote the same

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function). In Andrew's words: "with any x value, *excluding negative 2*, they would be equal to each other".

6.4 Dealing with the differences by exploring the values of the equivalent syntactic forms at the restriction: coming back to the numeric perspective

The interview continued with a sequence that involved the *Expanding technique with CAS* for finding the expanded form of a given expression (Part III: *Testing for equivalence by re-expressing the form of an expression—using the Expand command*). In this part of the interview, Andrew's use of CAS techniques was very relevant for exploring, confronting, and comparing the results obtained through the different perspectives on the algebraic equivalence of expressions.

Andrew typed in Expression 2, preceded by *Expand* and got the expanded form, as follows:

$$Expand[(x^2+x-20)(3x^2+2x-1)] \quad 3x^4+5x^3-59x^2-41x+20$$

Then, he obtained the expanded polynomial form of Expression 3: $3x^4+11x^3-25x^2-23x+10$, and anticipated the polynomial result he would get for Expression 5 by using the same *Expand* command. However, after having found the restriction for Expression 5, he was not quite sure:

> A: My prediction for this is that this [Expression 5] is going to be the same thing [as the expanded form of Expression 3]…because that's the result I've been getting the whole time, that they're equal to each other, except for the minus two. I don't know if that will change anything…

Using the CAS, Andrew obtained the expanded form of the polynomial corresponding to Expression 5: $3x^4+11x^3-25x^2-23x+10$. His previous work with paper and pencil corroborated the results he was obtaining with CAS at this moment. Yet, Andrew was looking for the effects of the restriction on the CAS expanded forms of the expressions, as is suggested by the following excerpt:

> A: I thought it was going to be the same, but I didn't know what would come out on the calculator, just because we figured out that, if it was minus two, then it wouldn't work, yeah, because then everything would be over zero.
> I: So, are you surprised that the calculator produced this [expanded form of Expression 5: $3x^4+11x^3-25x^2-23x+10$] for this particular expression [original Expression 5: $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$].
> A: No, but also, like, I don't know what to expect because it's just that one thing that wouldn't work I think, that one term that wouldn't fit in, minus two. Because it would be over zero [referring to Expression 5]. But it could be that somehow if you put in minus two, that this one [Expression 3] is zero too. So, it could be that my rule isn't necessarily correct, that the minus two wouldn't work.
> I'm saying that if the minus two was incorporated here [Expression 3], if I worked it out, like I could even do the, what's it called, what's that called the second…?
> I: The substitution key.
> A: Yeah, the substitution, like if I put minus two there [Expression 3], that might equal, like this [Expression 5] I know would equal zero… just 'cause it's over zero

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[Expression 5], but it could be that this one [Expression 3] equals zero too, expression 3, would equal zero as well, so that wouldn't even be right, the conjecture that I made.
I: What was that conjecture again?
A: That uh, minus… the only term [value of x] that wouldn't work for my theory that they [Expressions 3 and 5] were equal to each other, is minus two.
What I'm saying is that as far as I know, it is very possible that if I work this [Expression 3] out, with minus two incorporated into it, that that would equal zero too, which is based on the fact that they [Expressions 3 and 5] have always been equivalent, that the calculator didn't show anything.
I: Do you want to try? Test your theory with the calculator on number three.

At this moment, Andrew introduced the task of evaluating for $x = -2$ the different syntactically equivalent forms of Expression 5 (the original expression, the factored and simplified form, and the expanded form). Seemingly, Andrew was wondering what would happen when evaluating the equivalent expressions at the restriction: *Do they inherit the restriction from Expression 5?*

Andrew used the CAS to evaluate Expression 3 for $x = -2$ and got −84. Then, he proceeded to consider Expression 5. He had already found its domain restriction for $x = -2$ and the undefined value in that case. However, he was not sure about the result that he would obtain by using the CAS evaluation technique:

$$\left[\text{A types}\,\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}\,|x=-2\right]$$

I: Don't hit the enter button yet, what do you think is going to happen?
A: I think it's gonna be minus 84. [A hits the button and obtains "Undefined".]
A: Well that's what I think it is. That is just what I thought. I just didn't think the calculator would show it… I thought it was going to be like that, that's what I figured out it should be, undefined, but I didn't think the calculator would show it.

Then, Andrew evaluated the other syntactically equivalent expressions at $x=-2$. Table 6 shows the results obtained.

This task, which was not considered in the original design of the interview, could be described as a spontaneous confrontation of the establishment and corroboration of the syntactical equivalence of Expressions 3 and 5 (they denote the same rational fraction)

Table 6 CAS evaluation of the expressions (both rational and polynomial) syntactically equivalent to Expression 5 (at $x=-2$)

Expression		Value at $x=-2$ (using the CAS evaluation technique)
Original Expression 5:	$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	Undefined
Original Expression 3:	$(3x-1)(x^2-x-2)(x+5)$	−84
Expanded form of expressions 3 and 5, obtained by using CAS:	$3x^4+11x^3-25x^2-23x+10$	−84
Factored form of expressions 3 and 5, obtained by using paper and pencil:	$(x-2)(x+5)(3x-1)(x+1)$	(−84) Andrew considered this evaluation, but did not explicitly carry it out during the interview.

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Table 7 CAS evaluation for exploring the values of the forms that are syntactically equivalent to a given rational expression (whose simplified form corresponds to a polynomial), when it is treated as a rational function

Technique	Evaluate the rational expression at the restriction.
	Evaluate the corresponding expanded polynomial expression at the restriction.
	Evaluate the corresponding factorized and simplified polynomial expression at the restriction.
Technology	At the restriction, the rational expression is undefined; but the corresponding expanded and the factorized polynomial expressions are well-defined and their values are the same.
	The different syntactically equivalent expressions (rational expressions and polynomials) do not necessarily have the same domains of restriction.

Type of task: exploring the numerical values of the different syntactically equivalent expressions by evaluating them at the domain restriction of rational Expression 5

versus the fact that these expressions take on different values at the restriction (they do not denote the same function).

At the beginning, the restriction was incorporated only as an exception to the equality of the values of the expressions. But, seemingly at this moment of the interview, Andrew was exploring whether the restriction was inherited by the syntactically equivalent forms (i.e., the original expression, the factored and simplified form, and the expanded form) and, thus, if the equality of the values would remain for $x=-2$. In Andrew's words, "it is very possible that if I work this (Expression 3) out, with minus two incorporated into it, that that would equal zero too, which is based on the fact that they (Expressions 3 and 5) have always been equivalent."

The CAS evaluation technique allowed Andrew to realize that the restriction is not inherited through forms syntactically equivalent: both Expression 3 and the expanded and factorized forms of Expression 5 do not inherit the restriction; in fact, all three take the same real value: −84 (in fact, they denote the same function). The information being provided by the CAS seemed pivotal to the development of Andrew's understanding of the extent to which restricted values travel through the syntactic processes. The *epistemic value* of the CAS evaluation technique was central to Andrew's technological reflections on the equivalence of the expressions. In parallel with the solving of this spontaneous task (i.e., evaluating for $x=-2$ the different syntactically equivalent forms of Expression 5), Andrew was exploring this technique in itself, comparing it with the other techniques that he had used. The CAS evaluation technique allowed him to differentiate the syntactically equivalent expressions (equivalent rational fractions that include polynomials), when dealing with the restriction (i.e., treated as rational and polynomial functions). As Andrew said, "the calculator did show it." Table 7 presents our analysis of Andrew's spontaneous exploration of the

Table 8 Results obtained by applying the CAS equivalence test

Case	What the CAS displays
Expressions that are equivalent without any restrictions	True
Expressions that are equivalent with restrictions	True
Expressions that are not equivalent	The same equality between the two expressions that was entered at the input line

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effects of the restrictions on the forms syntactically equivalent to Expression 5, by using the CAS evaluation technique.

6.5 Making explicit the conflict between the two perspectives on equivalence and a first articulation of the differences

The interview continued with Part IV: *Testing for equivalence without re-expressing the form of an expression—using a test of equality.* For this part of the interview, we had designed some tasks that required the student to make explicit and confront the differences between the two perspectives on algebraic equivalence. For the solution of these tasks, we introduced two CAS commands: the *equivalence test* and the *numeric equality test.*

The *CAS equivalence test* allows one to discern whether two given expressions (considering the rational fractions that they denote, in fact) are syntactically equivalent. This test does not take into account the domain restrictions of the expressions (i.e., it does not takes into account the rational-function-related aspects). To apply this test, the two given expressions are typed into the calculator with an equal sign between them. The result is obtained by pressing the Enter key. For example, for Expressions 3 and 5, the equality introduced into the CAS calculator is: $(3x-1)(x^2-x-2)(x+5)=\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$, and the result obtained with the CAS is: True.

Table 8 shows the results that can be obtained by applying the CAS equivalence test to different expressions that can be equalized.

The *CAS test of numeric equality* evaluates the equality of the values taken by any two given expressions (considering the rational functions that they denote) evaluated for specific values of x. For instance, introducing the equality of Expressions 3 and 5, followed by the value for evaluating the expressions ($x=-2$ in this case): $(3x-1)(x^2-x-2)(x+5)=\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}\Big|x=-2$, produces the CAS result: False.

Table 9 shows the results that can be obtained by applying the CAS test of numeric equality to different expressions that can be equalized. In the case of expressions that are syntactically equivalent but which have a restriction (as for Expressions 3 and 5), the CAS results obtained by applying the equivalence and the numeric equality tests would seem to be "contradictory." For the restriction, when using the numeric equality test, the result that is obtained is "False" (the corresponding functions have different values there), whereas when using the equivalence test, the result is "True" (the corresponding rational fractions are equivalent). In working on the tasks of this part of the interview, Andrew was faced with this "contradiction." It was a critical moment for him, once more, one for which the CAS was again instrumental, semiotically speaking.

He typed the equality of Expressions 3 and 5 (i.e., the CAS equivalence test) and pressed Enter. The calculator returned "True":

Table 9 Results obtained by applying the CAS test of numeric equality

Case	What the CAS displays
Expressions that have equal values at the given value of x	True
Expressions that do not have equal values at the given value of x	False
Expressions that are both undefined at the given value of x	Undef=Undef

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A: That's what I expected… It [the calculator] is still not showing minus two.
Well, I don't know, I can't explain why all of this works, but when it's expanded it equals the same thing. So, I figure that, I don't know if it [the calculator] expands them first and then does it. It [the calculator] either expands them both to see if they are equivalent, maybe. Then it [the calculator] sees if they come out to be the same expression.

This result confirmed what Andrew had already obtained by applying the factoring and expanding techniques, with and without CAS. As Andrew said, this CAS technique neglects the domain restriction of Expression 5.

A: It probably… I figure that it [the calculator] knows we can put them in the same form, like it expands them so they are both in the same form, and you can tell, and when you expand them they come out to the same expression and you can't tell that the minus 2 doesn't work.

Then, by applying the numeric equality test for $x=-2$, Andrew obtained "False" and pointed out the following:

A: I had that before… there it is… negative 2. It [the calculator] realizes! It's not as smart as me… the calculator realizes that… Now the calculator realizes that the uh second, um expression, once it becomes over zero, that's undefined, and the first expression stays the same. Yeah, so it realizes it's false.

Andrew explained these contradictory results as follows:

A: Yeah, at first [he is referring to the result of the test of equivalence] it's saying that any value of x would be true, that any value of x can be substituted and they would be equivalent.
But, like this just proves, that when minus two is incorporated that it's not true [he is referring to the result of the test of numeric equality for $x = -2$], in this form at least [original expression 5]. Because once it's expanded, it [the calculator] saw they were still equivalent, and it didn't. I guess in different forms it's not true, but in this particular form [original expression 5] it is.

From our theoretical point of view, in this part of the interview Andrew constructed a first articulation of the differences and contradictions of the results obtained through the syntactic and numeric techniques: the numeric equivalence of two algebraic expressions could be established in a general manner (for every value of x, not just for a finite set of values) by means of syntactic techniques, using both paper and pencil and CAS. For example, through expanding and factoring techniques, Expressions 3 and 5 could be rewritten as the same expression. However, numeric equivalence requires considering the restrictions. At the domain restriction for Expression 5, these expressions do not have the same value.

This articulation allowed Andrew to provide an explanation for the contradiction between the results from the numeric equality and equivalence tests. Andrew explained the result obtained from the equivalence test by appealing to syntactic techniques, in his words: "it [the calculator] knows we can put them in the same form, like it expands them so they are both in the same form, and you can tell, and when you expand them they come out to the same expression [the same rational fraction]." But, as Andrew pointed out, it is necessary to consider the domain restriction of Expression 5. In Andrew's words: "when minus two is incorporated, it's not true [i.e., they are not the same rational function]."

Table 10 presents a theoretical analysis of the techniques and the technological discourses, both numeric and syntactic, that were articulated in the considerations and reflections made by

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Andrew during this part of the interview. These technological reflections take into account the double complication of handling different forms of expressions (factored and simplified and expanded forms) and dealing with both perspectives on algebraic equivalence: the numeric and the syntactic (considering the denoted rational function and the denoted rational fraction). It is noted that these conscious reflections on the part of Andrew came into being over time through a variety of semiotic means of objectification: through his interactions with the CAS and paper-and-pencil tools, as well as through the mathematical formulations, discourse, and particular expressions that were embedded within the tasks. Even if the role of the CAS is highlighted, these various other constituents played significant roles.

The interview continued, now touching upon a complementary aspect of algebraic equivalence: non-equivalence. The task was to apply the equivalence test to Expressions 2 and 3:

> Enter directly into your calculator's entry line the equation formed from the two given Expressions 2 and 3:
>
> $$\left(x^2+x-20\right)\left(3x^2+2x-1\right)=(3x-1)\left(x^2-x-2\right)(x+5)$$
>
> 1.What does the calculator display as a result?
> 2. How do you interpret this result?

Andrew anticipated that the result of the equivalence test would be "False":

Table 10 Andrew's numeric and syntactic perspectives on determining the equivalence of rational expressions and polynomial expressions

	Numeric perspective (the expressions denote rational functions)	Syntactic perspective (the expressions denote rational fractions)
Technique	Determine the restrictions, i.e., establish the common domain of the expressions. Compare the expressions over the common domain. Evaluate the expressions for several values of the common domain.	Rewrite the expressions in a common algebraic form, in a factorized or expanded form. Compare the rewritten expressions (term by term or factor by factor).
Technology	If the expressions take on the same values for a set of values, they can take on the same values for all values except for the restriction(s) (i.e., for any value of the common domain). Their *numeric equivalence* is conjectured.	If the expressions can be rewritten as the same expression (in a factorized or expanded form), they are *syntactically equivalent*.
Technology articulating both perspectives	Numeric equivalence of algebraic expressions can be established by means of syntactic techniques. However, at the restrictions, numeric equivalence does not correspond to the syntactic equivalence of the expressions; numerical evaluation is necessary in this case.	Numeric equivalence of algebraic expressions can be established by means of syntactic techniques. However, at the restrictions, numeric equivalence does not correspond to the syntactic equivalence of the expressions; numerical evaluation is necessary in this case.

Type of task: establishing the equivalence of polynomial expressions and rational expressions

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A: I think it's going to be false… Because of the results I've gotten before.
When expressions 2 and 3 were *expanded, they weren't equal*.
When I *factored* them out myself not using the calculator, *they weren't equal*.
When I *substituted in numbers, they weren't equal*.
[So] It would be, I believe it would be false.

Andrew had obtained the *expanded forms* and the *factorized forms* for Expressions 2 and 3 (considering them as polynomials), with paper and pencil (Part II) and by using the CAS (Part III), respectively. He had also found that these expressions (considering them as polynomial functions) took on different values when he evaluated them for a few values of x (Part I).

However, when he applied the CAS equivalence test to these expressions (by entering their equalization), the CAS returned the same equality. Very surprised at this result, Andrew looked for a mistake he may have made in the typing of the expressions:

A: I don't know what I did wrong.
A: It [the calculator] re-writes that… is that what's supposed to happen?
I: Yeah, actually that's the way it works.
A: I was expecting to get like "False" or "Undefined" or something like that… "Undefined"… not necessarily "Undefined" but something that showed me that it's not true, to say "False". I was expecting to see "False" as the answer, but I guess when it's re-written it just means that it's totally unreasonable!

Andrew tried to explain this CAS result that was obtained by applying the equivalence test to the equality of Expressions 2 and 3 (which he knew were not equivalent) by contrasting it against the obtained result for Expressions 3 and 5 (which were equivalent):

A: Basically [I mean] that they're not equal to each other.
So, what I assume the calculator does… is to expand them and put them in the same form. Like I said before, I assume the calculator expands them and then compares. That's why I was saying it was true for Expressions 3 and 5, it [the calculator] expanded them and that's why the $x+2$ doesn't come into consideration. So, I think that when it expanded them [referring to Expressions 2 and 3] it sees that they are in expanded form, but they're not equal to each other…

Andrew assumed that there were no values of x for which Expressions 2 and 3 took on equal values:

I: When you say equal each other, do you mean… equal regardless of the values of x?
A: I don't know, I don't know if for any value of x, they would be equal. I would assume not.

At this moment, the interviewer referred back to the initial part of the interview:

I: Well, this is a good time to refer back to the first page of the activity [Part I].
You said, if I understand you correctly, that when the calculator displays the same equation back, it tells you that it's never equal for any values of x. So…
A: But I… Yeah…
I: So if you go back here [table of values for Expressions 2 and 3], what is it… 2 and 3, for 2 and 3 are there any values that actually make them equal?

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A: Oh yeah, I didn't realize that! Yeah, $\frac{1}{3}$ and -5. So I would use the substitution button.

Andrew verified these results using the *numeric equality test*. He typed:

$$\left(x^2+x-20\right)\left(3x^2+2x-1\right)=(3x-1)\left(x^2-x-2\right)(x+5)|x=\left\{\frac{1}{3},-5\right\}$$

and, by pressing the Enter key, he obtained "True." He was not sure how to interpret this result. It is important to remark that by accepting this, it implies admitting that values of x do exist for which Expressions 2 and 3 take on the same value, even if "they weren't equal"! (i.e., they were not equivalent in terms of both the numeric and the syntactic perspectives). He said the following:

A: I would assume that, I don't know, obviously the calculator can do this, I guess, but I would assume it would come with the numbers, like I would think that we were saying that in some situations it is sometimes true. I would figure that it would tell you that…
As far as I was concerned I didn't really take note of this until we got further… As far as I was concerned these two [Expressions 2 and 3] weren't equal to each other no matter what, especially when the calculator showed me.

Andrew finally established his interpretation of the equalization of Expressions 2 and 3 as follows:

A: Expression 2 can be equivalent to Expression 3 when $\frac{1}{3}$ is substituted, but Expression 2 is not equivalent to Expression 3 just in general.
I: What do you mean by in general?
A: Like you're putting it in the same form. So, if they're equivalent, they should be the same, like identical, like 3 and 5 were and 2 and 3 aren't. So, like in general form they're not equivalent, but for certain numbers they are.

The tasks for this last part of the interview—though we include just the first of them in this analysis—are related with another issue, one concerning a specific use of the "equal sign" in algebra: the study of the theory of equations. To solve these new tasks, Andrew applied his mathematical knowledge and techniques for studying the equivalence of algebraic expressions. He found that two algebraic expressions could take on equal values for some specific values of x, even if they were not equivalent (neither syntactically nor numerically). By solving the tasks of this part of the interview, and by applying the equivalence and numeric equality tests, Andrew made explicit his resolution of the "contradiction" between the syntactic equivalence and the numeric equivalence of the given expressions. Also through these tasks, Andrew was able to begin to explore the meanings of some complementary aspects of equivalence: equations and their solution. In this later exploration, the syntactic and numeric techniques that were developed for studying equivalences were enriched and deepened to another level.

7 Discussion

The question that was central to this study was the following: *What is the nature of the technological discourse that students develop in the process of articulating the syntactic*

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(form) and numeric (functional) perspectives when grappling with the equivalence of polynomial and rational expressions in a task-based, CAS and paper-and-pencil, environment? In this article, we have described in detail the interview-supported experience of a 10th-grade student in the process of confronting and making sense of a particular cultural conceptual object, that of the double perspective on algebraic equivalence. The analysis highlighted the way in which Andrew's refining and gradual differentiating of his notion of the equivalence of polynomial and rational expressions unfolded over the course of his work with the designed tasks. The discussion of this concluding section of the paper will revisit what we believe were four pivotal moments of that process, and in discussing these moments we illustrate the central roles played by the interviewer, student, and cultural tools in making meaning for conceptual objects.

The cultural conceptual object that was the aim of this research on mathematical learning was the double perspective on algebraic equivalence, that is, the syntactic and numeric perspectives that underlie interpretations of equivalence in the study of polynomial and rational expressions. As we have pointed out, generally these two perspectives remain implicit or even confounded, the one with the other in school algebra. In the mathematical analysis of the theory underpinning this double perspective, which we presented above, we outlined distinctions between the two mathematical objects (rational fraction and rational function) that give rise to each of these perspectives and emphasized that while a syntactic perspective on equivalence holds when polynomial and rational expressions are viewed in one way (i.e., as rational fractions), a numeric perspective needs to be integrated in order to take into account possible domain restrictions when the polynomial and rational expressions are viewed as functions. Through the interview, as a result of his solutions and reflections on the proposed tasks and by means of the CAS and the paper-and-pencil tools and techniques, this cultural object was made "apparent" to Andrew (in the sense of Radford, 2006).

The CAS tool that was central to the design of the tasks for the study (i.e., the TI-92 Plus) is one that reflects the underlying mathematics in particular ways. Through the CAS techniques of expanding, factoring, and simplifying, the polynomial and rational expressions are treated syntactically. But they are also treated numerically, as when a CAS user, perhaps taking into account any restrictions, indicates an evaluation to be made with specific values. As was seen earlier, the *CAS Equivalence Test* involved the syntactic perspective, while the *CAS Numeric Equality Test* involved the numeric perspective. Thus, the CAS tool, which affords the possibility of considering both perspectives on equivalence, was a vital piece in the collection of signs and objects, the semiotic means of objectification to be used to make our mathematical intentions apparent (see Radford, 2003, 2006, for further theoretical treatment of *semiotic means of objectification*). As will be seen from the discussion that follows, this tool—a reflection of the mathematical cultural history of the topic under study—was the scarlet thread running throughout the process of Andrew's coming to differentiate and to articulate the two perspectives on equivalence.

As suggested by our analysis of the unfolding of the study, Andrew's initial approaches to testing for equivalence included formulating conjectures based on a small sample of numerical values, followed by a verification that involved the paper-and-pencil expanding technique. He brought this knowledge to the study with him. It was part of his past history. However, his techniques and technological discourses coexisted, without being explicitly differentiated. For Andrew, the applied techniques did not belong exclusively to a unique perspective. In fact, for him, differentiated perspectives on equivalence did not exist. They were just pieces of mathematical knowledge and resources for solving the same type of tasks: establishing the equivalence of algebraic expressions. Moreover,

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because he did not take any particular notice of the restriction of the denominator of Expression 5, his techniques and technological explanations led him to conclude that Expressions 3 and 5 were equivalent and equal for all values. Therefore, to provoke a first pivotal moment, the interviewer (it could have been the teacher in a classroom situation) decided on the spur of the moment to ask: "Is there any value of x that would not be permissible as a replacement value for x in Expressions 3 and 5?". When Andrew responded that he thought every number would work, it was suggested that he take note of the $x+2$ denominator in Expression 5 in thinking about the question that had been put to him.

That question, to which Andrew responded: "Negative 2, … But, well, I should have seen this before too. I don't know why I didn't," led to a cascade of significant actions on Andrew's part. He used the CAS tool to see what it would produce when he asked it to expand and simplify the rational expression that was Expression 5—wishing to test whether the CAS would in some way indicate the domain restriction of −2 that he now realized was an important part of judging the equivalence of Expressions 3 and 5. But the CAS did not indicate the restriction; it simply expanded and simplified the rational expression, producing the same result that Andrew had obtained with paper and pencil. We interpret this instance as being a second pivotal moment for Andrew. He came to realize that this mathematical tool (the CAS) carried out syntactic transformations with no heed to possible numerical restrictions.

So Andrew decided to try something else—in fact, he created a task for himself that had not been part of the designed task sequences. He spontaneously used the CAS evaluation tool to see what numerical result would be obtained if the CAS were to evaluate, at the restricted value of $x=-2$, the various syntactically equivalent forms for Expression 5 and the initial and expanded forms of Expression 3. He felt the need to test whether the restriction, which when substituted into the initial form of the rational Expression 5 had yielded the CAS result of "Undefined," would in some way be inherited by the other simplified forms of the rational expression. Clearly, they were not. The evaluations of the other syntactically equivalent forms at $x=-2$ had all yielded −84. This was a third pivotal moment in Andrew's coming to differentiate and to refine the two perspectives on equivalence. The rational Expression 5 did not yield the same value at the restriction as did the other expressions to which it was syntactically equivalent. So the idea that syntactical equivalence might not necessarily be the same as numerical equivalence was beginning to emerge.

The fourth pivotal moment, which reflects to a certain extent the role that the task design played in Andrew's coming to articulate the double perspective on equivalence, occurred soon afterward. With our introduction of the *CAS Equivalence Test* that yielded "True" when input with Expressions 3 and 5, and the *CAS Numeric Equality Test* that yielded "False" when input with the same two expressions and the evaluation of the x variable at −2, we had intended both to provoke a definitive confrontation regarding the equivalence of the polynomial and rational expressions and to spark attempts at explaining and thereby gradually refining the technological-theoretical discourse. The conflict produced by these two results brought the articulation of the two perspectives to the fore for Andrew. The conflict he faced had originated in the "contradictions" obtained by applying the syntactic techniques he knew (factoring and canceling, and expanding) for justifying the numeric equivalence of expressions (conjectured by the evaluating technique). For the restriction, his techniques and their corresponding technological discourses had given an "exception." However, to explain the contradictory results obtained by the CAS techniques *Equivalence test* and *Numeric equality test*, he had to re-elaborate his initial technological explanations of the relations

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between the results obtained through expanding, factoring and canceling polynomials and rational expressions and the results obtained through evaluating the corresponding polynomial and rational functions. Andrew articulated the differences and contradictions by establishing distinctions between the numeric and the syntactic techniques, as well as between their corresponding conceptual elements (technologies): *numeric equivalence* of algebraic expressions can be "proved" by rewriting them and showing that they are the same (*syntactic equivalence*). However, at the *restrictions* (numerical values of x where the rational expressions are not defined), numeric equivalence does not correspond to the syntactic equivalence of the expressions; numerical evaluation is necessary in this case.

Throughout this study, we have tried to show both the way in which Andrew came to make sense of the double perspective on algebraic equivalence that was at the heart of the study, and also the way in which this conceptual object *was made apparent* to him: by a complex interaction of designed task sequences, interviewer questions, CAS tool, and not least of all the student's own curiosity and spontaneous reflection and actions. Our analysis builds upon, and provides further evidence for, the theoretical approach developed by Radford (2006):

> Through words, artifacts, mathematical signs, and gestures—i.e., through semiotic means of objectification—the mathematical object … [was] made apparent to the students. In order to *see* it, the students underwent a process of objectification in the course of which their subjective meanings were refined. … Through the design of the lesson and the teacher's continuous interpretation of the students' learning, … the students' subjective meaning are pushed towards specific directions of conceptual development. (pp. 57–58)

We began the article by proposing that the tasks we designed for this study offer an arena in which it is possible to study the differences between "form" and "function" (Cuoco, 2002), that is, between the syntactic perspective and the numeric perspective. From our point of view, the presented results corroborate the importance of the gradual study of these relations and differences, which generally remain implicit or ignored but can constitute very important opportunities for conceptual reflection in school algebra. But our results also suggest how potentially useful a CAS tool—when carefully integrated into the design of the tasks—can be in the objectification of key algebraic concepts.

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Appendix

Equivalence of expressions

Part I (with CAS): Comparing expressions by numerical evaluation

I (A) The table below displays three algebraic expressions and two possible values for x.

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Using the two given values of x (i.e., $\frac{1}{3}$ and –5) and two others of your own choosing, calculate the resulting values for each expression by means of the evaluation tool of your calculator [i.e., the "with operator," (|)].

Important: Proceed one complete row at a time when filling in the table.

Record your choice of additional x values in the table's top row, and record the results in the appropriate cells below.

For $x=$	$\frac{1}{3}$	-5		
Expression	Result	Result	Result	Result
2. $(x^2+x-20)(3x^2+2x-1)$				
3. $(3x-1)(x^2-x-2)(x+5)$				
5. $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$				

I (B) Compare the results obtained for the various expressions in the table above. Record what you observe in the box below.

I (C) Reflection question:

Based on your observations with regard to the results in the table above (in I(A)), what do you conjecture would happen if you extended the table to include other values of x?

Part II (with paper and pencil): Comparing expressions by algebraic manipulation

II (A) Based on your observations in Part I A, make a conjecture as to which of the above set of given expressions might be re-expressed in a common form?

II (B) To test your above conjecture by means of paper and pencil algebra, re-express the given expressions below in another form (not the expanded form). Show all your work in the table's right-hand column.

Given expression	Re-expressed form of given expression
2. $(x^2+x-20)(3x^2+2x-1)$	
3. $(3x-1)(x^2-x-2)(x+5)$	
5. $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	

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II (C) In Part I C, you made some conjectures based on numerical evaluations of expressions. Explain in what way the algebraic manipulations in Part II B supported (or not) each of those conjectures.

For any conjectures of Part I C not supported by your algebraic manipulations in Part IIB, how do you account for the discrepancy?

Part III (with CAS): Testing for equivalence by re-expressing the form of an expression—using the EXPAND command

The left-hand column of the table below contains the expressions from the previous lesson. Using your calculator, fill in the right-hand column with the expression produced by the EXPAND command (see F2 menu in the calculator).

Syntax: EXPAND (*expression*)

Given expression	Result produced by EXPAND
2. $(x^2+x-20)(3x^2+2x-1)$	
3. $(3x-1)(x^2-x-2)(x+5)$	
5. $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	

Part IV (with CAS): Testing for equivalence without re-expressing the form of an expression—using a test of equality

It is possible to explore whether two expressions are equivalent without having to re-express their forms. An alternative approach is to use a CAS test of equality:

IV (A) Enter directly into your calculator's entry line the equation formed by expressions 3 and 5:

$$(3x-1)\left(x^2-x-2\right)(x+5)=\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$$

1. What does the calculator display as a result?

2. How do you interpret this result?

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3. Use your calculator's "with operator" (|) to replace x by –2 in the above equation. Interpret the result displayed by the calculator.

IV (B) Enter directly into your calculator's entry line the equation formed from the two given expressions 2 and 3:

$$\left(x^2+x-20\right)\left(3x^2+2x-1\right)=(3x-1)\left(x^2-x-2\right)(x+5)$$

1. What does the calculator display as a result?

2. How do you interpret this result?

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