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SOME IMPLICATIONS OF THE RESULTS OF A CASE STUDY FOR
THE DESIGN OF PRE-CONSTRUCTED, DYNAMIC GEOMETRY
SKETCHES AND ACCOMPANYING MATERIALS

ABSTRACT. The study that informs this article was undertaken to investigate the benefits and limitations of using pre-constructed, web-based, dynamic geometry sketches in activities related to deductive proof at the secondary school level. Two distinct themes emerged from analysis of the results – first, the relationship between the activities and the development of geometric thinking skills, and second, the relationship between the design of the materials, and the exploration process. This article focuses on the latter. An analysis of the data showed that task question and sketch provision must work together to create an environment for exploration. It also indicated that explicit attention to visual interpretation and exploration using change is required in order for students to benefit fully from their experiences with pre-constructed dynamic geometry sketches. This article draws attention to the idea that pre-constructed dynamic sketches (whether web-based or not) and accompanying materials are central elements of the learning activity of which they are a part, and therefore, that decisions about their design have the potential to support or impede the development of exploration strategies and geometric thinking skills. An examination of student responses to particular questions in light of visual and dynamic geometry research suggests that through the materials we may be able to improve students' use of the mathematical investigation process in exploring pre-constructed dynamic geometry sketches.

KEY WORDS: dynamic geometry, learning materials, mathematics education, pre-constructed sketch, task design, secondary school

When we say we educate children, it sounds like something we do to them. That's not the way it happens. We don't educate them. We create contexts in which they will learn.
(Papert, quoted in Pease, 1989)

The dynamic-geometry supported classroom offers a challenge regarding the creation of the contexts mentioned by Papert. Students in such classes may spend much of their class time interacting with a computer program, rather than communicating with a teacher. We hope that they will actively explore visual images, discuss, analyse, and communicate their findings. In order to ensure such focused and productive activity, we must develop worthwhile tasks that take into account the realities of this special learning environment.

Students can use dynamic software to construct and manipulate their own diagrams, but pre-constructed sketches are also available for use in geometry tasks. The study that informs this article was undertaken to in-



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investigate the benefits and limitations of using pre-constructed, web-based, dynamic geometry sketches in activities related to deductive proof at the secondary school level (Sinclair, 2001). Two distinct themes emerged from analysis of the results – first, the relationship between the activities and the development of geometric thinking skills, and second, the relationship between the design and use of the materials, and the exploration process. This article focuses on the latter.

BACKGROUND

Pre-constructed, web-based dynamic geometry sketches share some features with sketches pre-constructed with *Cabri Géomètre* (Baulac, Bellemain, and Laborde, 1992), or *The Geometer's Sketchpad* (Jackiw, 1991). Points on the sketch can be dragged to enable reasoning about invariant properties and to provide evidence about the validity of conjectures. Pre-set relationships, such as measurements and ratios, change to match the change that has occurred as a consequence of dragging. Web-based sketches also support action buttons to hide or show additional details, to move and to animate objects. Those created with *JavaSketchpad* (Jackiw, 1998) do not permit the user to construct or delete objects; however, those designed with Cinderella (1999, Richter-Gebert and Kortenkamp) can include options for constructing a limited number of objects.

Whether a pre-constructed sketch is web-based or not, it presents a geometric situation to the student in visual format. Since the creator of an image knows details that are hidden from an ordinary viewer, interpreting a pre-constructed sketch is similar to interpreting a picture that someone else has drawn. Dynamic sketches add the complexity of motion to this already difficult task. Thus, designing pre-constructed dynamic geometry sketches to support the development of mathematical understanding, involves understanding visual reasoning and its relationship to dynamic exploration.

Visual reasoning and dynamic exploration

Today more and more information is presented visually and research is showing that understanding and interpreting what we see – a process Wheatley (1998) calls ‘imaging’ – is not simple. According to Wheatley it involves three activities: constructing an image, re-presenting the image, and transforming the image. The implication is that the meaning we take from an image is not necessarily the meaning that someone else takes, because it depends in part on what we know about what we are looking at (p. 129).

One of the difficulties facing students is that mathematical pictures and diagrams contain a great deal of specific content information represented in a concise but 'nonsequential' format (Goldenberg, Cuoco and Mark, 1998). This requires students to apply their own organisation to the information, and to draw conclusions about how items are connected. The ability to interpret visual information is so important that Duval (1998) recommends that visualisation processes, including heuristic exploration of geometric problems, be developed separately from the construction processes and reasoning processes necessary for geometric analysis.

Behind heuristic exploration of geometric figures is the need to understand the role of invariants. Goldenberg, Cuoco, and Mark (1998) contend that before students are able to look for invariants in geometric relationships they must first develop "the ability to take apart in the mind, [and] see the individual elements" (p. 7). Presmeg (1999), noting that a diagram "by its nature depicts one concrete case or instance", suggests that to develop the ability to generalise, students need to have the opportunity to use 'more abstract' forms of imagery. In a 1986 study of visualisation in high school students, Presmeg found that dynamic imagery – although used by only a few 'visualisers' – was effective in this regard.

If examining relationships through motion is a technique that helps highly visual students, then we must determine whether, or how it can be taught. Teachers and textbook writers have always ensured that diagrams illustrate the generic case; however, when using dynamic software, a student can inadvertently create a special case by dragging. Students who have not learned to use change to explore a visual image, can only analyse a static onscreen sketch, regardless of its suitability as a model.

Ponte (2000) notes that the French, British, and Portuguese curricula, as well as the NCTM Standards, expect students to engage in investigative activities traditionally associated with scientific work, in order to formulate and test conjectures. Many teachers have incorporated dynamic geometry software in their programs because it provides an accessible and powerful tool for investigating geometric ideas. As Laborde (2001) notes, the "interplay of conjectures and checks, of certainty and uncertainty was made possible by the exploration power and checking facilities offered by the DG environment."

We know that to successfully explore a geometry problem with dynamic software, students must be able to verify, conjecture, generalise, communicate, prove and make connections (Chazan, 1990, p. 630). Even more basic however, this study showed that students must actually learn the process of exploration. In the most general sense, this means learning to notice, to pose questions, and, as mentioned earlier, to use change to

investigate relationships. The materials used are of central importance in helping students develop these skills. The findings of my study suggest that the materials 'intervene' in the learning environment and consequently must be designed with attention to several important criteria.

Interventions

In 1999 Towers researched the role of the mathematics teacher vis-à-vis individual students. She examined the nature of teachers' interventions and identified those that contribute to the growth of student understanding. Towers categorised the following intervention strategies: managing, checking, reinforcing, inviting, clue-giving, enculturating, blocking, modelling, anticipating, praising, retreating, and rug-pulling. She also identified three teaching styles: (a) showing and telling, (b) leading, and (c) shepherding. She found that the shepherding style and the strategies of rug-pulling and inviting consistently contributed to the growth of student understanding (Towers, 1999). Her definitions for these are:

Shepherding: An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing and prompting.

Inviting: Suggesting of a new and potentially fruitful avenue of exploration. More open-ended than clue-giving. [Clue-giving is a deliberate attempt to point the student to the correct answer or preferred route.]

Rug-pulling: A deliberate shift of the student's attention to something that confuses and forces the student to reassess what he or she is doing.

(Towers, 1999, pp. 200–202)

Towers' work is especially relevant for the technology-supported mathematics classroom because the nature of a lab classroom supports teaching small groups or pairs rather than delivering whole-class lectures.

In a recent study of the effects of technology-enhanced classrooms on students of low socioeconomic status, Page (2002) found an increase in students' self esteem, and significantly higher mathematics achievement – phenomena noted in several earlier studies (cf., Tyler and Vasu, 1995; Repman, 1993). He posits that the increase in student-centred interactions among students in the experimental group was an important factor in these results.

These two studies illustrate the importance of interventions and interactions within the learning environment. Extending these ideas further, I contend that the beneficial (and deleterious) effects of the technology-based activities in this study were strongly related to 'interventions' of the task materials themselves.

Design of task materials

In their 1996 summary analysis of research on computer-based learning environments in mathematics, Balacheff and Kaput note that one of the ways in which the computer makes its primary impact is by “changing the relationships between learners and the subject matter and between learners and teachers – by introducing a new partner” (p. 495). The resulting environment is didactically complex. In addition to the usual teacher-student interactions, there are interrelationships among the student, the computer and the task, which impact the learning situation (Balacheff, 1991; Sutherland and Balacheff, 1999). The task introduces a didactic framework, which permits students to interact with the particular software. The nature of this student-software interchange is, in turn, shaped by the unique characteristics of the software.

‘Task’ as used in this article refers to more than the mathematical problem or concept at its core. First, it includes the questions and instructions for students, whether onscreen or in printed form. These materials and teacher guidelines are fundamentally important as noted by Dreyfus (1991):

...the booklet that forms part of the environment includes a set of *worksheets* and a set of proposals how to integrate the software into class work. And it is these two components that make the software into a learning environment that can be implemented with average teachers in an average school system.

(Dreyfus, 1991, p. 127, italics in original)

However, in the case of pre-constructed sketches, the task also includes the sketch itself, together with any special investigative software tools created by the designer.

Keller, Wasburn-Moses, and Hart (2002) are presently investigating the impact of using on-line curriculum-focused applets to improve students’ visualisation skills with respect to isometric drawings. The team spent the first two phases of the project creating materials and developing a “robust applet for isometric drawing and spatial visualisation” (p. 18). This need for careful front-end attention to the design of materials is typical of any mathematics education project that seeks to use technological visual tools. It is partly related to the technical difficulty of constructing objects that will do what you want them to do; but it is also related to the following dilemma:

...everything that she undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion; if the teacher says what it is that she wants, she can no longer obtain it. (Brousseau, 1997 p. 41)

In designing a pre-constructed sketch and the accompanying materials, it is not trivial to determine which details to include or which questions to pose since making too much available can remove the motivation for exploring and providing too little can make a task impossible.

Since pre-constructed sketches present the student with collections of already assembled objects – as opposed to basic geometric elements with which to construct objects – accompanying materials should (at least in some areas) be fundamentally different from those developed for construction activities. This study examined how students responded to materials designed for use with pre-constructed, web-based sketches. The results provide insight into how the designer can exploit the potential interaction between two components of the task – the sketch and the lab sheet.

STUDY

The research used a case study approach and multiple sources of information – observation field notes, videotape, audiotape, a student questionnaire, and interviews with teachers. Collected data were transcribed, then analysed by coding, developing categories, describing relationships, and applying simple statistical tests where appropriate.

Participants

Sixty-nine students, aged seventeen to eighteen, in three classes – class A from school S1, and classes B and C from school S2 – participated in this study. All students were enrolled in a grade twelve advanced mathematics program that covered topics in algebra, geometry, analytic geometry and trigonometry. The study focused on the topic of congruence and parallelism, which was part of the geometry section (Curriculum guideline, 1985, p. 60). The students had done introductory work on deductive geometry related to this topic in grade 10 and on similarity in grade 11. None of the students had worked with dynamic software.

Students worked in pairs. In each class, several pairs were studied in more depth by audiotaping or videotaping their activities. The students in the taped pairs were chosen by the teacher to represent the range of achievement levels within the class, but students in any particular pair were not necessarily at the same achievement level. In the article, where appropriate, student aliases will be followed by indications of their achievement level (E, VG, G, A, or W, corresponding to Excellent, Very Good, Good, Average, or Weak) as determined by their teacher. For example, Pat and Dave, A/W, indicates that Pat's achievement was average and Dave's was

weak. In a transcript notation, Paul (E) indicates that Paul's achievement was excellent.

Materials

Four web-based dynamic geometry sketches were prepared for students to investigate during the sessions, two extra sketches were made available to students who finished early, and one sketch was prepared for a group discussion. The sketches included facilities for hide/show, movement, and animation where appropriate. Each sketch was constructed using *The Geometer's Sketchpad* and converted to html using *JavaSketchpad*. The html version maintained all relationships constructed in the original.

In the pre-study interview the three study teachers identified difficulties that their students experience in the geometry strand. For example, teachers mentioned that students constructing congruency proofs frequently select sides or angles that do not correspond to one another, or, in fact, do not even belong to the subject triangles. They noted that this problem usually occurs when figures overlap or are presented in rotated, reflected, or translated form. These student difficulties reveal an inability to 'see' each overlapping figure separately or to mentally transform a figure to a new orientation to compare it with another.

Such difficulties may stem from, or be associated with, the need to recognise a side or an angle as an element of two or more figures. Fischbein (1987) suggests that difficulties in this area are related to the *duplication obstacle*, identified by Duval (1983), which arises when students are faced with accepting that a mathematical object can play two roles. To address these difficulties, at the design stage I set up action buttons or provisions to highlight particular figures, to toggle details on and off, and to rotate or reflect shapes so that they could be superimposed, or viewed from the same orientation.

Problems chosen as the basis for the web-based sketches related to triangles and quadrilaterals and were similar in difficulty to those in the student text, *Mathematics: Principles and Process, Book 2* (Ebos, Tuck, and Schofield, 1986). Students were asked to prove triangles congruent, to prove altitudes equal, and to investigate the question, When do the diagonals of a parallelogram bisect one another?

Each lab sheet included: directions for opening and manipulating the sketch, a statement of the problem, and questions related to the task (see Appendix A for an example). Each question or instruction was designed to do at least one of the following:

- focus student attention on details in the sketch;

Prove: Triangle ABC congruent to Triangle FCB

▲ Show Given Information
 ▲ Hide

$m\overline{CF} = 1.6\text{ cm}$
 $m\overline{AB} = 1.6\text{ cm}$
 $m\angle ABC = 116^\circ$
 $m\angle FCB = 116^\circ$

→ Separate ABC and FCB
 ▲ Show reflection and mirror
 ▲ Hide Reflection
 → Match FCB and A'B'C'
 → Reset

▲ Show perpendicular through H
 ▲ Hide

Figure 1. Day 1, task 1 – View on selecting: ‘Show Given Information,’ ‘Separate ABC and FCB,’ and ‘Show perpendicular through H.’

- encourage students to explain their thinking about the relationships they observed;
- help students move through an investigation by prompting them to examine the evidence in the onscreen model, to check hypotheses, or to consider other possibilities;
- help students develop a proof.

Overview of session tasks

Three 75 minutes sessions or four 45 minutes sessions were held with each class. During this time, student pairs worked on four tasks. An additional task was done as a whole class activity. A very brief overview of the four main tasks and one of the extra tasks is included here to help the reader follow the discussion.

1. Day 1, task 1

This sketch was developed to introduce students to *JavaSketchpad*, and specifically, to address student difficulties with overlapping figures and selection of triangles.

In the sketch, $\triangle ABC$ and $\triangle FCB$ are reflections of one another in the perpendicular bisector of BC , which can be toggled on or off using the 'Show perpendicular through H' and 'Hide' buttons. When the sketch is first opened, point A is red, indicating that it can be manipulated. As point A is dragged, point F undergoes opposite motion as a reflection should.

The two triangles ABC and FCB can be separated using an action button with label, 'Separate ABC and FCB '. The button, 'Show reflection and mirror,' reflects triangle ABC in a red mirror line. The button, 'Match FCB and $A'B'C'$ ' causes triangle FCB to move on top of the reflection of triangle ABC , demonstrating congruency. A reset button is provided to move triangle FCB to its original position.

The 'Show Given Information' button controls the display of the markings that indicate the equality of AB and FC , and $\angle ABC$ and $\angle FCB$, as well as the measures of these lengths and angles. As a vertex of the figure is dragged, the measurements update.

The lab sheet for Day 1, task 2 (see Appendix A) asked students to prove $\triangle ABC$ congruent to $\triangle FCB$. The proof could be carried out by a straightforward application of the SSS (side, side, side) congruency theorem; the task aimed to help students select the correct sides by providing the tools to separate and reorient the triangles, and by asking specific questions about observations and results.

2. Day 1, task 2

The second task for day 1 was designed to address student difficulties with overlapping triangles, selection of triangles, and two-step proofs. All triangle pairs that could be selected were reflections, and congruency could be established or not established by considering what would happen if one member of the pair was flipped over. For example, triangle BEF and BDF can be shown to be congruent using SSS. This implies that BF is a line of reflection. If BD and EF are produced to intersect at A , then by the properties of reflection we know that BC is equal to BA and FC to FA . Since the curriculum focuses on Euclidean proof, study students were not able to compose a transformation argument such as this; however, their awareness of symmetry as shown later in this article suggests that they were ready to consider such an option.

This task inadvertently introduced an element of uncertainty, by including one pair of triangles (BAF , BCF) that could not be proven congruent *with the given information* (i.e., in two steps, although the congruency could be established through carrying out an additional congruency proof). This is not usually done in textbook problems.

Prove: $BA = BC$

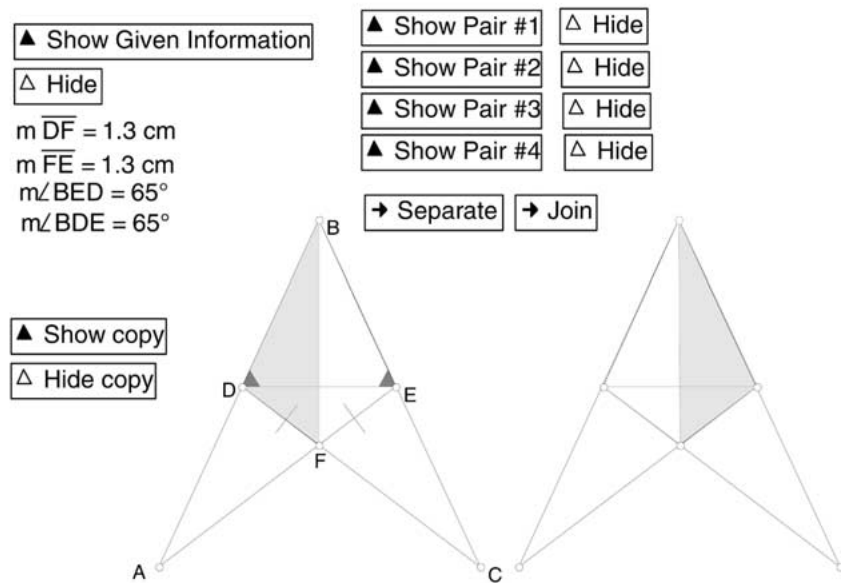


Figure 2. Day 1, task 2 – View on selecting: ‘Show Given Information,’ ‘Show Copy,’ and ‘Show Pair #1.’

To help students notice details and pick out shapes, the sketch included the following design elements:

- The four chosen pairs of congruent triangles (BDF, BEF; BAE, BCD; ADE, CED; and ABF, CBF) were shaded in four different colours; given equal angles were shaded red; information could be toggled off and on to further emphasise details; triangle pairs could be separated; measurements for the given angles and lengths were displayed; measurements updated as points on the sketch were dragged.
- Overlapping figures could be separated; colour was added to emphasise the shapes; colour was also used to overlay angles and sides within the shape.

It was expected that students would prove the pairs congruent in the following ways: BDF, BEF – SSS; ADE, CED – ASA; BDC, BEA – ASA. [Note: ASA refers to the angle, side, angle congruency theorem.] The lab sheet encouraged students to examine each pair by using the separate, join, and hide/show commands.

Prove: Triangle AMD is congruent to Triangle CNB

▲ Show Given Information
 △ Hide
 DC = 5.6 cm
 BA = 5.6 cm
 AD = 2.8 cm
 CB = 2.8 cm
 $m\angle BNC = 90^\circ$
 $m\angle AMD = 90^\circ$

▲ Show Triangle
 △ Hide Triangle
 → Move O →U
 → Move O →Midpt

Figure 3. Day 2, task 1 – View on selecting: ‘Show Given Information’, ‘Show Triangle’, and ‘Move O →U.’

3. Day 2, task 1

This task gave students the opportunity to apply properties of parallel lines and to investigate a problem using a rotation.

The triangles to be proven congruent were coloured to attract student attention. When a vertex of quadrilateral ABCD was dragged, AD and BC appeared to remain equal and parallel, as did AB and DC. When the ‘Show Given Information’ button was used, students could deduce that ABCD was indeed a parallelogram since opposite sides were marked equal.

It was expected that students would use ASA (angle, side, angle congruency theorem) to prove that $\triangle AMD$ and $\triangle BNC$ were congruent: $AD = BC$ (given), $\angle DAM = \angle BCN$ (parallel line law), and $\angle MDA = \angle NBC$; however, students could also investigate the relationship between the two by superimposing an additional given triangle over $\triangle ABC$ and then rotating it to fit over $\triangle CDA$. This movable triangle was a tool for testing whether $\triangle AMD$ and $\triangle BNC$ were congruent; however, it could also be used to demonstrate the fact that congruent triangles have congruent altitudes (i.e., $\triangle ABC$ and $\triangle ADC$ are congruent, which implies that AM must equal BN). Questions on the lab sheet such as: “What do you notice about the new triangle?” and “How can the information provided by these images be used to explain why $DM = BN$?”, were aimed at helping students notice and address the information provided in the sketch.

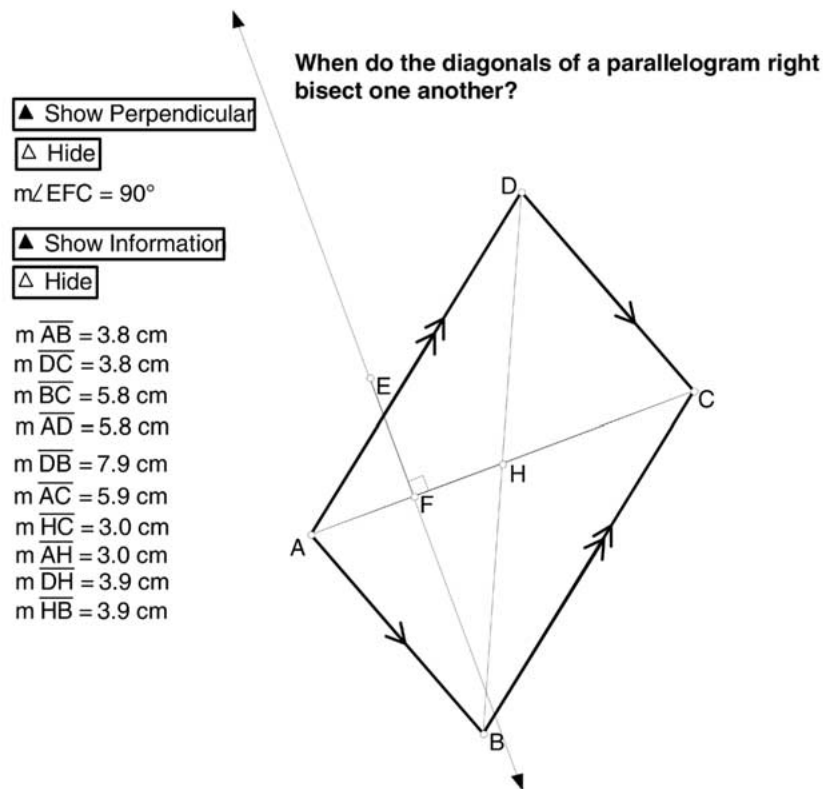


Figure 4. Day 2, task 2 – View on selecting: ‘Show Perpendicular,’ and ‘Show Information.’

4. Day 2, task 2

This task was designed to help students investigate the question: “When do the diagonals of a parallelogram right bisect one another?” The sketch included parallelogram ABCD, with diagonals AC and BD. The opposite sides were marked with arrows, the traditional markings for parallel lines. Measurements of the sides, diagonals, and semi-diagonals could be toggled on or off using Show/Hide buttons.

Since it can be frustrating to drag an angle until the measurement is precisely 90° , the button ‘Show Perpendicular’ produced a line perpendicular to AC. Students could drag the diagram until BD was aligned with the perpendicular – a slightly easier task.

Day 2, task 2 was undertaken after a short class session on conjecturing. Since the task involved knowing the term ‘right bisector’, the first section of the labsheet asked questions about the meaning of ‘bisect’, ‘right bisect’, and ‘right bisect one another’. The labsheet then asked students to drag the

Prove: Triangle CJL is isosceles

<input type="button" value="▲ Show Given Information"/>	<input type="button" value="▲ Show CN"/>
<input type="button" value="△ Hide"/>	<input type="button" value="△ Hide"/>
$m\angle CEG = 56^\circ$	<input type="button" value="▲ Show More"/>
$m\angle CGE = 56^\circ$	<input type="button" value="△ Hide"/>
$JE = 3.0 \text{ cm}$	
$LG = 3.0 \text{ cm}$	

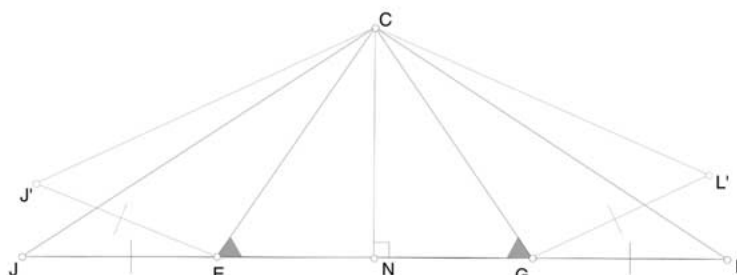


Figure 5. Day 1, task 3 – View on selecting: ‘Show Given Information,’ ‘Show CN,’ and ‘Show More.’

diagram and to conjecture a response, to develop a proof of the conjecture and then to outline an alternate proof.

5. Day 1, task 3

One of the extra tasks provided for students who finished early, Day 1, task 3, invited students to examine a sketch that looked like two nested isosceles triangles JCL and ECG. It asked the students to prove that triangle CJL was isosceles.

When the sketch was opened J, E, G, and L were collinear; however, students were able, using the Show More button, to examine a second situation in which these points were not collinear, and thus to consider why it is important to clarify initial assumptions. If the points are collinear, then the two triangles are isosceles. If the points are not, then the outer figure is not even a triangle.

Data coding and analysis

In the first run-through, I codified the transcripts according to what was happening at the time. Some examples of these first codes were: students on task, pointing to screen, dragging diagram, using motion button, deducting from visual, erroneous conclusion. The results were a rather disconnected series of observations.

On the second run-through, each of the annotated activities was examined to uncover the underlying motivation. Some of the codes at this stage were: checking understanding, modelling thinking, reinforcing ideas, posing inviting questions, colour used as reference.

As the work progressed I realised that many of the codes that described teacher interactions with taped pairs were similar to those described by Towers' (1999). I replaced codes (where appropriate) with hers and looked for evidence of the other styles and strategies that she had identified. Since I was analysing student pairs working together in an informal setting, rather than participating in a teacher-led lesson, I did find that certain interventions such as 'shepherding', which is a more subtle form of teaching, were more prominent – corroborating Towers' findings.

While the coding was beginning to highlight themes with respect to the teacher's role, most of the transcripts contained very little teacher-student discourse. There was more discussion between student partners. I developed new codes to describe these interactions: students correct one another, working at cross-purposes, struggle; however, it was obvious that students were reacting not just to their teacher or their partner but to some other stimulus.

I started to investigate the relationship between student comments and particular labsheet questions, or sketch features. First, I created an analysis table for each labsheet (see example in Appendix B). The first column included the instructions, statements, and questions from the original lab-sheet. The second column set forth the purpose I had intended that statement or question to fulfil. The third column described the role that I had envisioned for the sketch in relation to the statement or question. In choosing entries for the columns I used Towers' category labels wherever possible in order to facilitate comparisons to the transcripts, then developed additional descriptions as required. For example, beside 'Open Jsketch1', was the word 'managing' since this instruction was only intended to contribute to the management of the activities. Beside 'drag each red point and observe the diagram', the sketch column included the notation, 'draws attention – colour, movement.'

I then investigated student responses to these statements. In addition to new labsheet and sketch codes, which I inserted at appropriate questions and statements, new codes for student responses emerged, such as: hypothesises, struggles to remember, answers, is confused, concludes. These represented not just the outward actions but the inner thinking of the students, explicitly linked to the task components – the lab-sheets and the sketches.

DISCUSSION

Labsheet questions and statements were categorised as: managing, focusing, prompting, rug-pulling, inviting, and checking. Sketch provisions were listed as: providing a shared image, drawing attention through affordances such as colour and motion, supporting experimentation, and providing alternative paths. These categories helped me study the relationships between the sketches and the questions, and identify those that were significant influences on student explorations. Some were relatively uninteresting (such as instructions that managed – e.g., Open Jsketch1); others revealed important information about the learning environment.

The following analysis looks at the data regarding the study materials from two perspectives: 1) how students responded to particular question ‘types’ and sketch provisions; 2) how particular results highlighted elements that were missing (and needed) or poorly designed.

Noticing

Focusing statements such as the following, directed students to use particular affordances of the sketches. They – the question and the affordance – were intended to help students notice mathematical details.

1. Notice that some points in the sketch are red.
2. Click **Show Given Information #1**.
3. Click **Show Triangle**.

The following (unconnected) comments show that students at a range of achievement levels did notice details.

Doug (A): Angle BED – hey! . . . Angle BED is 72.455.

Katy (A): Um, uh the angle shadings. They’re the same angles. Yeah, I would say that. The angle shadings mean that they’re congruent angles. So, congruent sides and congruent angles.

Dave (W): They match. It matched it with the other one. It shows us that they’re congruent.

Paul (E): You’re working inside the thing again. Just look at – see the red part? Stare at the red parts. Blur out the black parts. No looking at the black parts – look at the triangle.

Colour drew attention – students noticed items that were coloured and sometimes missed those that weren’t. The ability to toggle colours off and on helped students to select and describe particular figures. Colour was also used as a simple and effective means of referencing objects in discussion as shown by Paul’s comments. On the other hand, despite Doug’s comment, the transcripts show that measurements were often ignored.

It is not clear whether students just did not notice the measurements – in *JavaSketchpad* lengths and angles are in a list and not attached to the object – or whether they were so attuned to deductive geometry that they did not expect to use measurement data. The ability to display an accurate image is commonly assumed to be a benefit of dynamic geometry software – it seems reasonable to conclude that the task of noticing and interpreting relationships between objects is easier if figures are drawn to scale. However, the study results showed that students do not automatically understand that the onscreen image is accurate.

Some students treated the onscreen image as if it were a pencil sketch – as if the diagram represented objects and their relationships, but was not drawn to scale. For example, they made comments such as: “Maybe cause it’s slanted you can’t tell it’s a square” (Barb and Clara, VG/VG, March 29, 2000), and “If this is equilateral these sides would have to be equal” (Paul and Sue, E/E, March 28, 2000). [In this latter instance, the triangle was clearly not equilateral]. I hypothesised that such responses might stem from prior use of textbook diagrams. Geometry teachers frequently warn their students not to make conclusions based on the appearance of these diagrams because they are not necessarily accurate.

Whatever the reason, the tendency of study students to gloss over measurements is in stark contrast to their awareness of colour and motion. It is of concern because the ability to explore using change requires focused attention to details that update under the operation of dragging.

Taking action

Once students have focused their attention on a particular object they need to do something meaningful. Particular questions and instructions prompted students to take action. For example:

1. Explain the meaning of the tick marks and the angle shading.
2. Write two additional facts that you know and explain why they are true.

All students responded to this type of question/instruction by examining the sketch and discussing their answers. The pre-constructed web-based sketches supported their investigations by providing dragging capabilities, onscreen data, and familiar labels and marks. The next two excerpts show how the affordances were used by students in response to prompting statements.

1. In response to, “Use your observations and the appropriate congruency theorem (SSS, SAS or ASA) to prove that $\triangle ABC$ is congruent to $\triangle FCB$ ” (Day 1, task 1, see Figure 1):

<p>Katy: They're the same. We're still looking for a third. OK reset it. OH, see how they're sharing a side? It has to be equal.</p> <p>Bea: BC?</p> <p>Katy: BC. Cause they share the side--so it has to be the same length.</p>	<p><i>Uses reset button to view triangles together</i></p>
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(Katy and Bea, A/A, April 3, 2000)

2. In response to "What shape is ABCD? Explain." (Day 2, task 1, see Figure 3):

<p>Pat: It's a rectangle – ABCD</p> <p>Dave: This equals that and this equals that. [He points to each side.]</p> <p>Dave: Ya. AD is equal to BC, right? Right?</p> <p>Pat: AD is equal to BC</p> <p>Dave: Is that given? I think that's given.</p> <p>Pat: Let me see it separated, OK?</p>	<p><i>Uses the static diagram as a reference</i></p> <p><i>Looks for evidence</i></p> <p><i>Uses separate button</i></p>
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(Pat and Dave, A/W, March 29, 2000)

Although initially intrigued by the ability to drag points, the study results show that students usually stopped dragging after a short time and concentrated on interpreting the static figure. I believe this indicates several things: 1) that the provision of the dragging capability is not enough to help students interpret pre-constructed dynamic figures and that in order to make effective use of dynamic diagrams students must be able to direct their use of dragging, i.e., they must learn to use change to explore; and 2) that task designers must focus on developing questions specifically about the motion.

Surprise

One question which was only *intended* to prompt students to gather evidence for congruency generated a great deal of discussion and investigation. The question was:

Do you have 3 pieces of information to show whether each pair of triangles is congruent?
(Day 1, task 2, see Figure 2)

In fact, there were not always three pieces of information – a situation that created uncertainty and surprise. It was this 'rug-pulling' question that first made me aware of the strong relationship between students and materials.

Research undertaken by Hadas and Hershkowitz (1999) demonstrated that uncertainty helps students develop understanding of geometric con-

cepts. When students are unable to construct an example using dynamic software to confirm an apparently reasonable conjecture, uncertainty results. This is similar to Towers' rug-pulling intervention through which teachers deliberately confuse students to draw their attention to subtle relationships. The actual question does not usually introduce uncertainty. In most cases, the student considers that a particular question is easily answered, but upon further investigation the student realises that the result is not straightforward.

In the study tasks, there were several examples of questions that led to uncertainty when the accompanying sketch was explored.

Example 1:

Day 1, task 3, one of the extra sketches (see Figure 5), was deliberately designed to surprise. Initially it looked like a simple nested set of isosceles triangles. However, the base could be broken and thus, the nature of the exterior figure should not have been assumed. Unfortunately because of time constraints only two taped pairs worked on this task. In the following excerpt, Joe and Bob come to a new awareness. In response to, "Is it true that $\triangle CJ'L'$ is isosceles?" (Day 1, task 3, see Figure 5) they have the following discussion:

Bob:	Um. Isosceles?	
Joe:	Sure. But it – but it might not be – like – like Bob, it could not be an isosceles. Say I went like this. . .	<i>Experiments with sketch</i>
Bob:	Look – L N J.	<i>Surprise</i>
Joe:	Ya, but what do you need that for? You don't need that for anything.	
Bob:	Oh, you can need it.	
Joe:	You know what the answer is? No they're not isosceles because if you move one they both have to move. Is it isosceles right now? I don't think so.	<i>Concludes correctly that it is not isosceles</i>

(Joe and Bob, A/A, March 30, 2000)

Example 2:

The second example that led to uncertainty was not intentional. As noted earlier I included pairs of triangles in Day 1, task 2 (see Figure 2) that did not have three pieces of information available to prove them congruent (and since the sketches were web-based they could not construct or measure anything else). This confused students because their experience of geometry problems was limited to situations that could be solved. Students

reacted to this surprising situation in several ways. Some spent a great deal of time – certain that they must be missing something. Others made up information! For example:

- Sarah and Earl decided that there were three pieces of information because of the ‘bisector theorem’. They returned to the problem several times. Sarah can be heard asking, “Is the bisector allowed?” She didn’t question their decision to include BF as a bisector – just worried that they might not be allowed to use it. At one point she said, in a rather frustrated voice, “I feel obligated that there must be 3 pieces of information given” (Sarah and Earl, G/A, April 4, 2000).
- Lily and Fran couldn’t find anything more than the fact that BF was a common side. They asked the next group for help, but were not told anything. They re-examined the sketch – separating it again and again. Fran attempted to include in the proof, relationships of sides and angles inside the triangle but Lily corrected her, saying: “We’re not supposed to be focusing on that”. After I led them through an organised check, they realised that they didn’t have three pieces of information, but they weren’t confident enough to abandon the search until I arrived (Lily and Fran, VG/A, April 4, 2000).

These episodes underscore the importance of circulating among students throughout sessions and of gathering students together to discuss ideas and conclusions. Moreover, while surprise can be a powerful way to engage students, if this surprise is initiated by a dynamic image – especially one that the student did not construct—in a non-teacher directed activity, materials should incorporate some question or statement that hints at the possibility of a surprising result to avoid causing undue frustration for students.

Inviting

Inviting questions such as the following, were more open-ended than the prompting questions mentioned earlier.

1. What additional information can you deduce about point H from the diagram? (Day 1, task 1, see Figure 1)
2. How can the information provided by these images be used to explain why $DM = BN$? (Day 2, task 2, see Figure 3)

These questions asked students to use their observation and interpretation skills, and to look at the problem from a different perspective. To answer, students needed to explore alternative paths.

The need for alternatives highlights an important design consideration. In order for students to explore uncharted territory, a pre-constructed web-

based sketch must provide options. Nevertheless, providing options is only part of the issue. All of the study sketches could be dragged into configurations that went well beyond the needs of the particular assignment. Measurements of various sides and angles were provided, even when they were not strictly required. And although students usually carried out traditional congruency proofs, all sketches could be explored via transformation relationships. However, many students did not know how to make use of these provisions. I contend that this problem had nothing to do with their computer skills and everything to do with their inability to pose their own questions.

Exploring involves making observations, then making choices about what one will do next. Decisions are based on answers to questions – “What will happen if I pull this point?” “How are A and B related?” “Is M a midpoint?” The designer of task materials cannot put all these questions on the screen or on a lab sheet – like Brousseau’s teacher who does not want to reveal too much, the designer must ensure that the task is open enough to allow the student to develop his or her own understanding. Instead, to help students move through an investigation the task creator can include 1) general exploration directives and suggestions, especially those related to using motion, and 2) statements that encourage students to pose their own questions about visual information.

Transformations and visual ideas

Student responses to another ‘inviting’ question in the study draw attention to the fact that we often focus on methods that are more suited to symbolic rather than visual analysis. Day 1, task 1 (see Figure 1) included the following question: “How can the information provided by these images be used to explain why $\triangle ABC$ is congruent to $\triangle FCB$?” Here, the students were asked to prove the two triangles congruent – not in the usual way by “trolling for triangles” (Whiteley, personal communication, 2000), but by applying their observations of images that were controlled by onscreen buttons. Many students showed that they appreciated the significance of reflections in this sketch; nevertheless they were unable to construct a satisfactory transformation-based explanation in answering the question. For example, Katy states:

No but listen – the reflection – show the reflection. . . . This triangle FCB is a reflection of ABC. Therefore it has to be equal – equal because it’s a reflection.

(Katy and Bea, A/A, April 3, 2000)

Pat and Dave, working on the same problem have the following conversation:

Pat:	It breaks it apart. It helps you to see it better.	<i>Reference to visual</i>
Dave:	This is the mirror	<i>Notes reflection</i>
Pat:	It breaks it apart so it's easy to picture it.	
Dave:	They match. It matched it with the other one.	<i>Understands congruency</i>
Pat:	It shows us that they're congruent. . . .	
Dave:	The mirror effect shows that ABC's just like FCB.	

(Pat and Dave, A/W, March 28, 2000)

In another example, a sketch allowed students to use rotation to explain why two segments were equal, instead of deducing the result via a triangle congruency proof. The question was: "How can the information provided by these images be used to explain why DM equals BN?" (Day 2, task 1, see Figure 3). Students intuitively understood that when the triangle was rotated, DM would fall on BN. Clara briefly commented: "Because the triangle fits – the triangle fits both," however, she did not follow up with a step-by-step analysis. She felt that the result needed no further explanation.

These impoverished conversations about transformations highlight students' unfamiliarity with describing visual information in precise terms. The study students had worked on symmetry and transformations in earlier grades, but most of the emphasis had been on recognising and reproducing transformed shapes; the development of deductive proof had avoided transformations completely. The experience of the study students is not atypical. Whiteley (1999) contends that many elementary and secondary school programs ignore transformational reasoning leaving students unprepared for the demands they will face in today's visual world.

When we notice that the motion of two points suggests they are rotations or reflections of one another, we are using transformation relationships to organise dynamic visual information. Since transformations can help us make sense of what we are viewing, they should be emphasised in any dynamic task. However, questions and sketch must complement one another. In the study sketches, transformation capabilities, while interesting to students, were not as effective as I had hoped in helping students develop their geometric skills because labsheet questions – and the students' curriculum – emphasised Euclidean-based methods. If we want to encourage transformation-based approaches to working with dynamic sketches methods must be developed to help students: 1) learn to use transformation concepts to monitor how visual elements are changing in relation to one another, and 2) develop facility in writing transformation-based proofs.

Explaining

Every labsheet included questions that asked students to explain their reasoning, but responses were usually sparse. For example, in answering “Find another pair of congruent triangles in the figure. Explain your reasoning” (Day 1, task 1, see Figure 1) three students wrote the following:

Example 1:

$\triangle GBH$ and $\triangle GCH$ are two new congruent triangles that are formed. The perpendicular line forms a common side. [Note: the view in Figure 1 does not show point G. When the reflected triangles merge, point G is the intersection of AC and A'C'.]

Example 2:

$\triangle ABG \cong \triangle FGC$ because
 $\angle ABC = \angle FCB$ given [Note: this angle isn't in $\triangle ABG$]
 $AB = FC$ given
 $\angle AGB = \angle FGC$ VOAT [Note: Vertically Opposite Angle Theorem]

Example 3:

$\triangle BGH \cong \triangle CGH$
 Proof: $GH = GH$ common
 $BH = HC$ right bisector
 $BG = CG$ ITT [Note: Isosceles Triangle Theorem]

When inviting questions asked for an explanation, students wrote more; however, comments were always very general in nature. For example, in response to “How can the information provided by these images be used to explain why $\triangle ABC$ is congruent to $\triangle FCB$?” (Day 1, task 1, see Figure 1), three students wrote as follows:

Example 1:

Both triangles have uneven sides (scalene). From the reflection in the mirror they are both symmetrical. When $\triangle FBC$ is placed upon $\triangle ABC$ they cover one another exactly.

Example 2:

The buttons do reverse images and try to place the triangles together. They fit so they are congruent.

Example 3:

It breaks it into two diagrams so it is easy to picture it. The mirror effect shows that it is the same triangle. It shows that it is exactly the same.

This tendency – to write very little, or to substitute traditional congruency proofs for explanations suggest that the study students needed more practice developing explanations of their mathematical reasoning. However, it is interesting to examine these answers from another perspective. I asked students to ‘explain’ in order to check their understanding of the concepts. Another very powerful incentive for an explanation is the need to communicate how you explored – to focus on process. Since students must learn how to interpret visual information, and how to use change to explore, perhaps this type of explanation is more valuable.

Avoiding trivial responses

Some questions on labsheets were more effective than others. For example, the question, “What do you notice about the new triangle?” was too vague. It resulted in conclusions such as the following:

Clara: It gets bigger and smaller.

Barb: It moves according to how you – It changes.

(Barb and Clara, VG/VG, Day 2, task 1, see Figure 3, March 29, 2000)

To avoid trivial responses, task developers can make questions more pointed. However, closing the boundaries of a question often leads to a decline in the cognitive demand of the task. The study students did offer trivial responses, but I contend that instead of a simplification of the mathematics, they needed explicit work on visual reasoning skills to help them focus on details, follow chosen elements across the screen, and use transformation concepts and other mathematical principles to organise information.

Sketch-Labsheet links

Study results show that the materials used in a geometry investigation involving pre-constructed sketches strongly impact the student. If we examine each statement or question in a task in terms of its function we find that some closely parallel the interventions used by a teacher. For example, some instructions ‘manage’ as a teacher would, some questions check for understanding, and some invite, by asking students to investigate, explain,

and explore more. I contend that labsheet statements also *prompt*, a category that suggests guidance but avoids the idea of adaptation implicit in a teacher's response. Questions and instructions also *focus student attention* on various aspects of a sketch such as a measurement, a relationship between sides or angles, or behaviour under dragging.

In order to facilitate exploration a sketch must provide the means to address the questions and instructions. In general:

1. When a question aims to focus student attention, the sketch must provide the visual stimulus. It must *draw attention* through colour, motion, and markings.
2. When a statement prompts action, such as asking students to drag, observe or deduce, the sketch must contain the necessary provisions. It must *provide affordances* so that the student can take the required steps.
3. Questions that invite exploration are open-ended. In order to explore uncharted territory, the student requires a sketch that allows options. Thus, when a question invites exploration, the sketch must *provide alternate paths*.
4. A question can *surprise* – which may lead to further exploration; however, the teacher is not necessarily there to correct any misinterpretation. Thus, the sketch must *support experimentation* to unmask the confusion. It must be flexible enough to help students examine cases, yet constrained enough to prevent frustration.
5. Questions that check understanding are important parts of any learning situation. In the study tasks, the checking involved students looking together for the answer. Although peer-interactions were not discussed in this article, study results showed that the sketch aided this process by *providing a shared image* for students to consider and discuss.

An important underlying thread must be a focus on provisions that will help students learn how to use change to explore, and how to extend their visual interpretation skills.

CONCLUSION

The activities for the study were designed to help students notice geometric details, explore relationships, and develop reasoning skills related to geometric proof. I intended to focus solely on the mathematics but was drawn by the data into an analysis of the interactions within the learning environment between the students and the materials. In terms of the materials I initially considered the sketches to be far more important than the

labsheets. However, an analysis of the data showed that task question and sketch provision must work together to create an environment for exploration. In addition, student responses clearly indicate that explicit attention to visual interpretation and exploration using change is required in order for students to benefit fully from their experiences with pre-constructed dynamic geometry sketches.

I do not want to leave the reader with the impression that task materials somehow act on their own in the learning environment. My aim in this article was to draw attention to the idea that pre-constructed dynamic sketches (whether web-based or not) and the accompanying materials are central elements of the learning activity of which they are a part, and therefore, that decisions about their design have the potential to support or impede the development of exploration strategies and geometric thinking skills. An examination of student responses to particular questions in light of visual and dynamic geometry research suggests that through the materials we may be able to improve the context in which students learn the mathematics of dynamic geometry.

APPENDIX A

*Day 1 – Task 1**Jsketch2*

1. Open Jsketch2.htm as described in the general instructions.
2. Notice that some points in the sketch are red. They are ‘draggable.’
 - Drag each red point and observe the diagram.

Tip: If the sketch gets really messed up just click Reload in the tool bar.
3. Click Show Given Information. Drag point A again and observe the diagram. Explain the meaning of the tick marks and the angle shading.

4. Click each button on the top right. How can the information provided by these images be used to explain why $\triangle ABC$ is congruent to $\triangle FCB$?

5. Use your observations and the appropriate congruency theorem (SSS, SAS or ASA) to prove that $\triangle ABC$ is congruent to $\triangle FCB$.

6. Click Show Perpendicular through H. What additional information can you deduce about point H from the diagram?

7. Find another pair of congruent triangles in the figure. Explain your reasoning.

APPENDIX B.
 Day 2, task 1: Question analysis form – early coding (see Figure 3 for sketch)

Instructions	Labsheet	Sketch
1. Open Jsketch1.htm	Managing	
2. Drag each of the red points A, B, and C and observe the diagram.	Focusing attention – visual	Drawing attention – colour, movement
3. Click <input type="button" value="Show Given Information"/> . Drag the red points again and observe the measurements. What shape is ABCD? Explain.	Focusing attention – detail	Drawing attention – hide/show; Drawing attention – colour, movement
4. Use your conclusions and the appropriate congruency theorem (SSS, SAS or ASA) to prove that $\triangle AMD$ is congruent to $\triangle CNB$.	Checking for understanding Enculturating – mathematics; Prompting action	Providing shared image Providing shared image
5. Click <input type="button" value="Show Triangle"/> .	Focusing attention – visual	Drawing attention – colour, movement
• Drag point B. What do you notice about the new triangle?	Prompting action Focusing attention – visual Checking for understanding	Supporting experimentation Drawing attention – colour, movement
• Rotate the new triangle using Handle. What do you notice?	Prompting action Focusing attention – visual Checking for understanding	Supporting experimentation Drawing attention – colour, movement Providing shared image
6. Click on Move O->Midpoint. [It's easier if you have AC in a horizontal position.]	Managing; Focusing attention – visual	Drawing attention – colour, movement
• Line the Handle point up with point C. What is the relationship between the new triangle and $\triangle ABC$?	Focusing attention – visual; Checking for understanding	Drawing attention – colour, movement Providing shared image
• Use the Handle to rotate the triangle about point O. What is the relationship between the new triangle and $\triangle ADC$?	Focusing attention – visual; Checking for understanding	Drawing attention – colour, movement Providing shared image
1. How can the information provided by these images be used to explain why $DM = BN$?	Inviting exploration	Providing alternate paths

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