Activity 6: Factoring

# Point of insertion: Chapter 3 of the « Reflections 436 » textbook, p. 184, after students have studied the identities involving the sum and difference of cubes.

**Objectives:** To establish connections between notions that students already know, across the application of these formulas:

;  ; ,

and to develop the generalization given by: .

# Part I (Paper & pencil and CAS, 20 minutes): Seeing patterns in factors

1. (a) Before using your calculator, try to recall the factorization of each algebraic expression listed in the left column of this table :

|  |  |
| --- | --- |
| Factorization using paper and pencil | Verification using FACTOR (show result displayed by the CAS) |
|  |  |
|  |  |
|  |  |
|  |  |

### Classroom discussion of Part I, 1a, followed by the

### simultaneous student work and discussion on 1b and 2a-2d

Reflection on the results obtained in 1a. We wish to consider these same identities, but from a different point of view. Toward that end, let us look at the factored forms of  and  to see if we can detect some signs of a pattern. “What do you notice about these two?” Among the expected student responses, some should be related to the fact that (*x*− 1) is a factor of the polynomials  and . Helping students to understand components of the newly emerging pattern of these factorizations can be encouraged by having them relate explicitly both the factored and expanded forms of these two examples. We suggest that the teacher write on the board the following, while asking:

 \_\_\_\_\_\_\_\_\_\_\_ and  \_\_\_\_\_\_\_\_\_\_\_\_\_

“So what do we obtain when we multiply out these factored expressions?”

Ask each student to work out the corresponding responses on their accompanying activity sheets (Question 1b). By performing the indicated multiplication, students should arrive at the same expanded forms that they began with in Question 1a. Questions 2a to 2d can then be done as a whole classroom group, with students sharing their responses and their underlying rationale as they go through each one.

We obtain (students would also write below on their sheets):

1. (b) Perform the indicated operations:

|  |
| --- |
|  |

|  |
| --- |
|  |

2. (a) Without doing any algebraic manipulation, anticipate the result of the product.

2.(b) Verify the anticipated result above using paper and pencil (in the box below), and then using the calculator.

|  |
| --- |
|  |

Note to teacher: It is important to discuss the idea that some terms get cancelled.

2. (c) What do the following three expressions have in common? And, also, how do they differ?

, , and .

2. (d) How do you explain the fact that the following products result in a binomial: two binomials, a binomial with a trinomial, and a binomial with a quadrinomial?

\*\*\*\*\*\*

**Classroom Discussion following Question 2d**

Discussion: At this point, the class can address the different methods of multiplication, making the link to the telescoping of the terms in the resulting products.

For example, with the distributivity method of multiplication:

,

|  |  |
| --- | --- |
| and with the vertical form of the multiplication algorithm: |  |

2.(e) On the basis of the expressions we have found so far, predict a factorization of the expression.

2. (f) Explain why the product (*x* –1) (*x*15 + *x*14 + *x*13 + … + *x*2 + *x* + *1*) gives the result *x*16–*1*?

Remark: The issue is to know whether, at this point, students are able to anticipate the resulting expression in relation to the cancellation of terms due to multiplication.

2. (g) Is your explanation (in (f), above) also valid for the following equality:

(*x* –*1*) (*x*134 + *x*133 + *x*132 + … + *x*2 + *x* + *1*) = *x*135–*1* ?

Explain:

\*\*\*\*\*

# Classroom discussion of Part I

The results we obtained above can help us anticipate a general factorization of the expression , for integral values of *n*. The teacher could pose the question:

“Can we anticipate the result before using the calculator?”

At this point, the discussion can address the importance of the factor  and the fact that in multiplying this binomial by any polynomial of the form ,

the result is the sum of the first and last terms of the product. This result is obtained by distributing the factor *x*, then the factor –1, throughout all the terms of the second polynomial, thereby leading to the pair-wise cancellation of all “inner” terms.

In sum, for integral values of *n* we have



\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

## Part II (40 minutes): Toward a generalization (activity with paper & pencil and with calculator).

II(A) 1. In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

|  |  |  |
| --- | --- | --- |
| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

II.(A).2. Conjecture, in general, for what numbers *n* will the factorization of :

1. contain exactly two factors?
2. contain more than two factors?
3. include  as a factor?

Please explain:

**Classroom discussion of Part II A**

Before asking about their conjectures, the discussion should touch upon the different ways of factoring x6 – 1, that is, an expression where *n* is a multiple of both 3 and 2.

Conjectures: We anticipate that students will first distinguish the cases of *n* even and *n* odd. We anticipate that many of them will (incorrectly) conjecture that there are more than two factors for even numbers *n* > 2 and only two factors when *n* is odd.

The teacher may wish to offer  as a counterexample to this conjecture, for which the factored form contains three factors.

We recommend that, at this point, students work either individually or in groups in order to revise their conjectures. They will work on Part II B.

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**Part II continued (with paper and pencil, and with calculator)**

II(B) 1. As with Part A above, each line of the table below must be filled in completely (all three cells), one row at a time, before proceeding to the next row. Start from the top row and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

|  |  |  |
| --- | --- | --- |
| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

II(B).2. On the basis of patterns you observe in the table II.B above, revise (if necessary) your conjecture from Part A.

That is, for what numbers *n* will the factorization of :

i) contain exactly two factors?

ii) contain more than two factors?

ii) include  as a factor?

Please explain.

II(C) Without using your calculator, answer the following questions:

1. Does 

* 1. contain more than two factors?
  2. include  as a factor?

Please explain:

2. Does 

1. contain more than two factors?

ii) include  as a factor?

Please explain:

3. Does 

1. contain more than two factors?

ii) include  as a factor

Please explain:

**Classroom discussion of Part II B and C**

# Whole-class discussion

The aim of Part II has been to incite students to discover the general relationship

, for integral values of *n*.

More specifically, we aim to promote the realization that for *n* prime, the factorization will be of this form; while for *n* even, the form of the complete factorization, which involves a refactorization of the second factor, will include both (x – 1) and (x + 1) as factors.

In the discussion following Part II A, the counter-example



aimed at promoting distinctions among the cases of *n* prime, even *n* > 2, and odd *n* not prime.

**If *n* is even and *n* > 2**, then  and  are two factors of the factored polynomial.









**If *n* is a prime number**, the factored expression contains only two factors:











**\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

**Part III: Challenge**

Explain why (*x* + 1) is always a factor of  for even values of *n* ≥ 2.

|  |
| --- |
|  |

**Optional** **Investigation**

**(note that these two pages are not included in the Student Version)**

**Note to teacher:** If we wish to promote the study of complex roots of polynomials that we have developed to this point, we might have students do an investigation that addresses ideas we wish to develop further. For this investigation, students will need to know how to multiply two complex numbers.

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Considering the different polynomial expressions we have worked on up to this point and that our goal was to find the roots of those polynomials, we now turn to studying what we call the **« roots of unity »**.

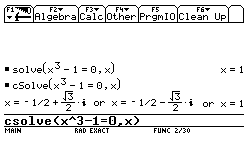
For example, to calculate the real roots of the polynomial equation , which is equivalent to *x*4 = 1, we can proceed via the factorization:

.

This factorization leads to the solutions , ,  and, which are four roots in the complex plane.

|  |  |
| --- | --- |
|  |  |

Another example: finding the real roots of the polynomial *x*3 – 1 reduces to solving the equation , which can be done using the calculator’s SOLVE command. But we can also find real and *complex* roots of this polynomialby searching for complex solutions of  using the CSOLVE command of our calculator:



This is the same as saying that the equation  is equivalent to

.

Note that ,  and *x3* =. These three complex numbers,

*x*1, *x*2 and *x*3, called « *the three third roots of unity*», are the three solutions of the equation , which is equivalent to *x*3 = 1. Thus, the solutions are third roots of 1.

These three complex numbers have the property that they are located on the unit circle (i.e., the circle centered at (0,0) with radius of length 1) in the complex plane:

|  |  |
| --- | --- |
|  |  |

In fact, *x*2 and *x*3 are roots of the polynomial *x*2 + *x* + 1. That is, they are solutions of the equation *x*2 + *x* + 1 = 0. Moreover, a product of any such complex roots always produces one of these roots.

Can we continue in this same manner with the other expressions we have developed up to now?

Did you notice certain regularities in obtaining roots of unity?