Using CAS in symbolic algebra at the secondary level : a classroom activity

André Boileau
Denis Tanguay

UQAM
Département de mathématiques
Section didactique

Contributors to this research

- Carolyn Kieran, Paul Drijvers, José Guzmán, Fernando Hitt, Ana Isabel Sacristán and Luis Saldanha.
- Our gratitude to:
  - the teachers and students of the participating schools,
  - our project consultant, Michèle Artigue,
  - the Social Sciences and Humanities Research Council of Canada, and the Ministère des Relations Internationales, who have funded, and continue to fund, this research.

CAS USE IN SECONDARY SCHOOL MATHEMATICS CLASSES

- Students' CAS use:
  - considered quite appropriate in college-level mathematics courses,
  - but not so much the case for secondary school maths up to now.
- In the past, many secondary school maths teachers have preferred developping paper-and-pencil skills in algebra to using CAS technology (NCTM, 1999).
- However, these attitudes are changing, based on:
  - research findings;
  - leadership of teachers and mathematics educators (and their impact on curricula and ministerial decisions);
  - greater availability of resources for using this technology at the Grade 9, 10, and 11 levels of secondary school.

BUT, WHAT DOES SOME CAS RESEARCH SAY?

- In France, since the mid-1990s:
  - CAS made their appearance in secondary maths classes.
  - Researchers (Artigue et al., 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the technical dimensions in algebra learning.
  - This emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996).
- From their observations, the research team of Artigue and her colleagues
  - came to think of techniques as a link between tasks and conceptual reflection
  - inferred that the learning of techniques was vital to related conceptual thinking.

Tasks-Technique-Theory

- Our research group
  - was intrigued by the idea that algebra learning might be conceptualized in terms of a dynamic among Task-Technique-Theory (T-T-T) within technological environments;
  - began, from 2002 up to this day, a series of studies that explored the relations among task, technique, and theory in the algebra learning of Year 10 students (15-16 years of age) in CAS environments.

Technique and theory emerged in mutual interaction: Techniques gave rise to theoretical thinking; and the other way around, theoretical reflections led students to develop and use techniques.

(Kieran & Drivers, 2006)

CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE?

We propose that it includes being able:

- to see a certain **form** in algebraic expressions and equations, such as a linear or quadratic form;
- to see **relationships**, such as the equivalence between factored and expanded expressions;
- to see through algebraic transformations (the technical aspect) to the underlying changes in form of the algebraic object;
- to explain/justify these changes.

CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE

Some classic examples of conceptual understandings in algebra include:

- the distinctions between
  - variables and parameters,
  - identities and equations,
  - mathematical variables and programming variables, etc.
- both the knowledge of the objects to which the algebraic language refers (generally numbers and the operations on them) and certain semantic aspects of the mathematical context, so as to be able to interpret the objects being treated…

Some more examples

1. Conceptualizing the equivalence of expressions in several forms (factored, expanded, etc)

   e.g., awareness that the same numerical substitution (not a restricted value) in each step of the transformation process will yield the same value:

   $$(x+1)(x+2) = x(x+2) + 1(x+2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$$

   and so substituting, say 3, into all four expressions is seen to yield the same value, here 20, for each expression.

CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE

2. Coordinating the 'nature' of equation solution(s) with the equivalence relation between the two expressions,
   e.g., for the following task,

> Given the three expressions
>
> $$x(x^2 - 9),\ (x+3)(x^2 - 3x) - 3x - 3,\ (x^2 - 3x)(x+3),$$
>
> a) determine which are equivalent;
> b) construct an equation using one pair of expressions that are not equivalent, and find its solution;
> c) construct an equation from another pair of expressions that are not equivalent and, by logical reasoning only, determine its solution.

CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE

3. Seeing the underlying forms through symbols, e.g.,

   (a) $x^6 - 1$ as $((x^3)^2 - 1)$ and as $((x^2)^3 - 1)$,

   and so being able to factor it in two ways.

   (b) x^2+5x+6 and x^4+7x^2+10

   seen both of the form ax^2+bx+c.

How Year 10 students in our project drew conceptual aspects from their work with algebraic techniques in a CAS environment

- The study
  - Tasks created by the researchers and proposed to teachers
  - Class sessions and interviews of students and teachers on video
- Concerning the tasks:
  - The tasks went beyond merely asking technique-oriented questions;
  - The tasks also called upon general mathematical processes that included:

    observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying.
- Concerning the technologies:
  - Both CAS (*TI-92Plus*) and paper-and-pencil were used, often with requests to coordinate the two.
  - The CAS provided the data upon which students formulated conjectures and arrived at provisional conclusions.

Example of such a task

- Factoring expressions of the form $x^n - 1$

  (adapted from Mounier & Aldon, 1996)

- Aim: to arrive at a general form of factorization for $x^n - 1$ and then to relate this to the complete factorization of particular cases for integer values of n, from 2 to 13.

One of the initial tasks of the activity

1. Perform the indicated operations: $(x-1)(x+1)$; $(x-1)(x^2+x+1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product
   $(x-1)\left(x^3+x^2+x+1\right)=$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ?
   $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)\left(x^3+x^2+x+1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. Is your explanation valid for the following equality:
   $(x-1)(x^{134}+x^{133}+x^{132}+\ldots+x^2+x+1)=x^{135}-1$? Explain.

$$(x-1)\left(x^{n-1}+x^{n-2}+\cdots+x+1\right)=x^n-1$$

Factoring $x^n - 1$ (case n=2 to n=6)

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Example of a student's work

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

$$x^4-1=\left(x^2\right)^2-1=\left(\left(x^2\right)-1\right)\left(\left(x^2\right)+1\right)=(x-1)(x+1)\left(x^2+1\right)$$

Factoring $x^n - 1$ (case n=2 to n=6)

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
New strategy after working with n=4 ?		
$x^5 - 1 =$		
$x^6 - 1 =$		

Factoring $x^n - 1$ (case n=7 to n=13)

Conjecturing (after factoring from *n*=2 to *n*=13)

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain.

Search for pattern

$$x^{ab} - 1 = \left(x^a - 1\right)P_{b-1}\left(x^a\right)$$

$$= \left(x-1\right)P_{a-1}\left(x\right)P_{b-1}\left(x^a\right)$$

where $P_k(y) = y^k + y^{k-1} + \cdots + y + 1$

(x+1) factor of (x^n – 1) iff n is even

Possible arguments

- Using (x – a) factor of P(x) iff P(a)=0
- Using

$$\begin{aligned} x^{2k}-1 &= \left(x^2\right)^k - 1 \\ &= \left(x^2-1\right)\left(\left(x^2\right)^{k-1}+\left(x^2\right)^{k-2}+\cdots+\left(x^2\right)+1\right) \\ &= \left(x-1\right)\left(x+1\right)\left(\left(x^2\right)^{k-1}+\left(x^2\right)^{k-2}+\cdots+\left(x^2\right)+1\right) \end{aligned}$$

- Other arguments ?

n = even number

$$x^8 - 1$$

$$(x-1)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x +$$

$$x^6(x+1) + x^4(x+1) + x^2(x+1) + 1(x+1)$$

THE ROLE OF THE TEACHER

- Are good tasks and CAS technology all that are needed to render technique conceptual, that is, to develop a conceptual understanding of algebraic technique?

  It would seem not !

- Another crucial ingredient is the teacher's *orchestration* of classroom activity that gives rise to the conceptualizing of technique in technology environments.
- Our present research project : characteristics of teachers' classroom practice involving CAS technology that relate to drawing out the conceptual aspects of technical work in algebra.

THANK YOU !

Questions?

Discussion?

References

See the associated Web Page:

http://www.math.uqam.ca/_boileau/ACA2009.html